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Groups definable in o-minimal and NIP settings

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Preface

The aim of this work is to present, in a concise and self-contained manner, some of the main results in the study of o-minimal groups, as seen from the point of view of NIP theories. In particular, we present a complete proof of the compact domination conjecture for definably compact groups, based on the recent work of Simon on NIP theories [Sim14].

For a long time, model theory concentrated mainly on the study of stable structures, i.e. structures that cannot define an order. During this period, Shelah first isolated the notion of NIP theories as a possible well-behaved generalization of stable theories. However, in the beginning, those theories were not studied per se, and only some basic properties were established. It was only many years later, in [PS86], that Pillay and Steinhorn, building on the work of Van den Dries [vdD98], introduced a new class of well-behaved unstable structures, the o-minimal structures, that has been an active area of research ever since.

It was noted early that the study of o-minimal theories, and especially of o-minimal groups, had many similarities with the study of stable theories. It was also known from the beginning that both o-minimal theories and stable theories are particular cases of NIP theories. However, it is only with the recent introduction of the notion of Keisler measures that a connection could be established between these facts.

The study of Keisler measures is connected with the efforts to understand groups definable in o-minimal structures. Pillay showed early in [Pil88] that o-minimal groups always admit a unique topology making them similar to real Lie groups. Since then, understanding which properties of Lie groups generalize to definable groups has been a major line of research. Unfortunately, in most cases, the absence of an exponential map and the presence of infinitesimal elements make it impossible to apply methods from Lie groups theory to the study of definable groups.

A major development in this direction was Pillay's conjecture that every group G definable in an o-minimal structure has a canonical way to define a "subgroup of infinitesimal elements" G^{00}, and that the quotient G/G^{00}, endowed with the logic topology, is a compact real Lie group of the same dimension of G [Pil04]. The beauty of this conjecture is that it makes an unexpected connection between the pure lattice of type-definable subgroups and the Euclidean structure of a canonical quotient of the group.

The first part of Pillay's conjecture was proved shortly after by the work of Berarducci, Otero, Peterzil and Pillay [BOPP05], while the part about the dimension was proved some years later by Hrushovski, Peterzil, and Pillay in [HPP08]. This last proof introduced the use of Keisler measures in the study of definable groups and identified the previously little considered NIP theories as a natural setting in which to study these questions. At the same time, in [HPP08], they introduced a new conjecture, called compact domination, which questions the possibility to lift the Haar measure of G/G^{00} to a Keisler measure on G. The compact domination conjecture was harder to solve, and a proof was found only some years later in [HP11]. It should be noted that this conjecture has important consequences on the study of the topology of o-minimal groups [Ber06] [BM11].

After its first introduction in [HPP08], the study of NIP groups and Keisler measures has developed into an interesting and mature theory. Thanks to this theory, we are now able to see that, if we generalize the fundamental notion of a type with that of a Keisler measure, then many important results on the study of stable groups also apply to o-minimal and NIP groups. Furthermore, a proof of the compact domination conjecture for suitable groups in NIP theories has been found recently [Sim14]. From a model theoretic point of view, this proof is more satisfactory than the previous one, as it does not make any use of external tools.

In this work, we want to review the major results in the study of o-minimal groups in the light of this new theory. At the same time, we want to use this occasion to present a concise and self-contained exposition of some of the most interesting and difficult aspects of the theory of o-minimal groups. For this reason, a great deal of attention attention has been paid in using the recent advancements in the subject to eliminate any unnecessary requirement from the original proofs, and to rewrite them using the most recent model theoretic point of view.

In the first part of this work, we introduce NIP theories and we develop the major tools we will use later, in particular Keisler measures. Still working in the NIP setting, we prove, for any definable group, the existence of G^{00}. Moreover, we prove compact domination for fsg-groups under suitable hypotheses. This part is mostly based on the work of Pillay, Hrushovski and Simon (cf. [Sim14]).

In the second part, we concentrate on o-minimal theories. First, we prove the main structure theorems for definable sets and we talk about point-set topology in o-minimal structures. Then, we turn our attention to o-minimal groups. The main theorems in this part are the descending chain conditions for type-definable subgroups and the fact that definably compact groups are fsg-groups. The former tells us that the quotient G/G^{00} is a compact real Lie group, while the latter, in conjunction with the results of the first part, proves that definably compact groups in o-minimal theories are compactly dominated. The proof of these two theorems is not as easy as one could hope, and we first need to introduce some structure theorems for definable groups.

Building from this work, we hope we will be able to derive a large part of the theory of o-minimal group directly from the general theory of NIP groups. We reserve to add results in this direction, that were not included here, in a future publication with Alessandro Berarducci.

Prerequisites and conventions

Since this work wants to provide a self contained introduction to groups definable in o-minimal and NIP theories, we do not assume any previous knowledge of those topics. However, we do assume some previous knowledge of the basic notions of model theory, in particular concerning compactness, quantifiers elimination, and indiscernible sequences. The reader can refer to any book on model theory, such as [Mar02, Chapters 1-5], for an introduction to these topics.

The conventions we adopt are mainly standard, nonetheless, it could be useful to recall them in order to fix the notation. We work with a complete theory T in a language L. As usual, we have a monster model $\mathcal{U}$ that is κ-saturated, for κ some inaccessible cardinal. We say that a subset of $\mathcal{U}$ is *small* if its size is strictly less than κ. We usually denote with M a small submodel of $\mathcal{U}$.

Let $A \subseteq \mathcal{U}$ be any set. We denote by $L(A)$ the set of formulas with parameters from A. Often, we write $\phi(x, b) \in L(A)$, meaning that $\phi(x, y)$ is an L-formula and that the parameters b are taken from A. We usually make no distinction between elements and tuples. Hence, by x we mean a tuple of variables $x_1, \ldots, x_n$, where $|x| = n$ is the size of the tuple. Similarly, for any subset $A \subseteq \mathcal{U}$, we write $a \in A$ when we really mean a tuple $a \in A^{|a|}$. When it is important to distinguish between elements and tuples, we write tuples with a bar over them.

Sometimes, in a formula, we want to distinguish between variables and parameters variables. In such cases, we talk of a *partitioned formula* $\phi(x; y)$ and we separate the variables x from the parameters y using a semicolon. If $\phi(x, b) \in L(\mathcal{U})$ is a formula and $A \subseteq \mathcal{U}$ is any subset, we put $\phi(A, b) = \{a \in A^{|a|} : \mathcal{U} \models \phi(a, b)\}$. We usually identify a definable set with the formula that defines it. Therefore, if X is the set defined by the formula $\phi(x, b) \in L(M)$ and $A \subseteq \mathcal{U}$ is any subset, we let $X(A) = \phi(A, b)$. Often, instead of writing $X(\mathcal{U})$, we just write X.

A *partial type* $\pi(x)$ over A is any coherent set of $L(A)$-formulas. A *complete type* over A, or simply a *type* over A, is a maximal coherent set of $L(A)$-formulas. We denote the space of types over A by $S(A)$. A *global type* p is a complete type over $\mathcal{U}$.

Recall that $S(A)$ is a compact Hausdorff space when endowed with the Stone topology. We say that a complete type p concentrates on a definable set $X = \phi(x, b)$ if $p \vdash \phi(x, b)$. The set of types over A concentrating on X is denoted by $S_X(A)$. Notice that $S_X(A)$ is a closed subspace of $S(A)$.

Let I be any linearly ordered set, by an *indiscernible sequence* over A indexed by I we mean a sequence of tuples $(a_i : i \in I)$ such that for any two increasing tuples $i_1 < i_2 < \ldots < i_n \in I$ and $j_1 < j_2 < \ldots < j_n \in I$, and for any formula $\phi(x, b) \in L(A)$, we have $\phi(a_{i_1}, \ldots, a_{i_n}) \leftrightarrow \phi(a_{j_1}, \ldots, a_{j_n})$.

Now, let $(a_i : i \in I)$ be any sequence. We define the Ehrenfeucht-Mostowski type, or EM-type, over A of the sequence to be the set of $L(A)$-formulas $\phi(x_1, \ldots, x_n)$ such that $\mathcal{U} \models \phi(a_{i_1}, \ldots, a_{i_n})$ for all $i_1 < \ldots < i_n \in I$, $n < \omega$. Recall that, by Ramsey theorem and compactness, for any sequence $(a_i : i \in I)$ there is an indiscernible sequence indexed over I and having the same EM-type.

Chapter 1

NIP theories

While NIP theories are fundamental for this work, we won't try to give here a complete exposition of this topic. Rather, we refer the reader to [Sim14], from which most of this chapter is derived. Our purpose here is to make this work as self-contained and easily readable as possible by giving a concise exposition of the main tools we will use later.

The first section of this chapter defines the concept of a NIP formula and build some of the theory we will need to prove that o-minimal theories are NIP. Historically, NIP formulas where introduced by Shelah in the early years of model theory while studying the link between the size of a set and the number of types over it. For many years, however, there was little interest in them, and even if it was noticed from the beginning that o-minimal theories were NIP, this fact did not find any appllication until the recent proof of the compact domination conjecture for definably compact groups in o-minimal theories [HPP08] [HP11].

In particular, this proof introduced the use of Keisler measures, the main topic of the rest of this chapter. Basically, Keisler measures are fintiely additive measures on definable sets. If one thinks of types as a 0-1 measures, as we encourage the reader to do, then Keisler measure can be seen as their natural generalization. We now understand that many nice properties of types in stable theories are not lost in the NIP context, simply they translates to properties of measures. A large part of the study of definable groups in NIP structures depends on the study of their invariant measures, however this will be the topic of the next chapter.

1.1 (Not the) Independence Property

Definition 1.1. Let $\phi(x, y) \in L$ be a formula. We say that $\phi(x, y)$ is NIP if we cannot find an indiscernible sequence $(a_i : i < \omega)$ and a tuple $b \in \mathcal{U}$ such that

$$\models \phi(a_i, b) \iff i \text{ is even.}$$

We say that a theory T is NIP if every formula is NIP.

An alternative and useful characterization of NIP formulas is given by the following.

Proposition 1.2. *A formula $\phi(x;y)$ is NIP if and only if for every $b \in \mathcal{U}$ and for every indiscernible sequence $(a_i : i \in I)$ there exists an end segment of $J \subseteq I$ such that the truth value of $\phi(a_i, b)$ is constant for $i \in J$.*

While the definition we have given is often the most useful in model theory, NIP formulas can also be characterized, as it was originally done, using the notion of *VC-dimension*. Let $\phi(x;y)$ be a partitioned formula. We say that a set A of $|x|$-tuples is *shattered* by $\phi(x;y)$ if we can find a family $(b_I : I \subseteq A)$ of $|y|$-tuples such that

$$\mathcal{U} \models \phi(a;b_I) \iff a \in I, \text{ for all } a \in A.$$

We call VC-dimension the maximal integer n such that ϕ shatters a set of size n. If there is no such integer, we say that the VC-dimension is infinite. Notice that, by compactness, ϕ has infinite VC-dimension if and only if it shatters an infinite set.

Proposition 1.3. *A formula $\phi(x;y)$ is NIP if and only if it has finite VC-dimension (equivalently, if it does not shatters an infinite set).*

Proof. ($\Leftarrow$): Suppose that $\phi(x;y)$ does not shatter any infinite set and suppose by contradiction that there is an indiscernible sequence $(a_i : i \in \omega)$ and an element $b \in \mathcal{U}$ such that $\phi(a_i, b)$ holds if and only if i is even. Now, let $I \subseteq \omega$ be any subset of ω. Clearly, we can find a strictly increasing map $\tau : \omega \to \omega$ such that, for every $i < \omega$, $\tau(i)$ is even if and only if $i \in I$. As $(a_i : i < \omega)$ is indiscernible, the function that maps $a_i \mapsto a_{\tau(i)}$ is a partial isomorphism. Let ψ be a global extension of this partial isomorphism, and take $b_I = \psi^{-1}(b)$. It is easy to verify that b_I has the required property.

($\Rightarrow$): Suppose by contradiction that $\phi(x;y)$ shatters an infinite set $A = \{a_i : i \in \omega\}$. By considering an indiscernible sequence having the same EM-type, we may assume that the sequence $(a_i : i < \omega)$ is indiscernible. Let $E \subset \omega$ be the subset of even integers, then there is some $b_E \in \mathcal{U}$ such that $\phi(a_i, b_E)$ holds if and only if $i \in E$, that is, if and only if i is even. $\square$

To prove that a theory is NIP, it is often useful to reduce to a simpler class of formulas. The following statements provide a way to do so.

Lemma 1.4. *A formula $\phi(x;y)$ is NIP if and only if $\phi^{opp}(y;x) = \phi(x;y)$ is NIP.*

Proof. By compactness, we can assume that $\phi(x;y)$ shatters the set $A = \{a_i : i \in P(\omega)\}$ witnessed by tuples $b_I \in \mathcal{U}$, where $I \subseteq P(\omega)$. Let $I_j = \{X \subset P(w) : j \in X\}$. Notice that $\phi(a_i, b_{I_j})$ holds if and only if $j \in i$. This shows that $\phi^{opp}(y;x)$ shatters the infinite set $(b_{I_j} : j \in \omega)$. $\square$

Lemma 1.5. *A boolean combination of NIP formulas is NIP.*

Proof. It follows directly from the definition that the negation of a NIP formula is NIP. Now, consider a conjunction $\phi(x;y) \wedge \psi(x;y)$ of NIP formulas (the disjunction case is similar). By Proposition 1.2, to prove that $\phi(x;y) \wedge \psi(x;y)$ is NIP it sufficies to show that for any indiscernible sequence $(a_i : i < \omega)$ and tuple $b \in U$ the truth value of $\phi(a_i, b) \wedge \psi(a_i; b)$ is definitely constant. As $\phi(x;y)$ and $\psi(x;y)$ are both NIP there is a some $N < \omega$ such that both $\phi(a_i, b)$ and $\psi(a_i, b)$ are constant for $i > N$. Hence, $\phi(a_i, b) \wedge \psi(a_i; b)$ must be constant for $i > N$. $\square$

Proposition 1.6. *Assume that all formulas $\phi(x; y)$ with $|y| = 1$ are NIP. Then T is NIP.*

Proof. Under these hypotheses, we prove the following claim.
Claim: Let $(a_i : i < |T|^+)$ be an indiscernible sequence of tuples, and let $b \in \mathcal{U}$, $|b| = 1$. Then there is some $\alpha < |T|^+$ such that the sequence $(a_i : \alpha < i < |T|^+)$ is indiscernible over b.
Proof: Suppose that for every $\alpha < |T|^+$ the sequence $(a_i : \alpha < i < |T|^+)$ is not indiscernible over b. Then for some formula $\delta_\alpha(x_1, \dots, x_{k(\alpha)}; y)$ we have

$$\models \delta_\alpha(a_{i_1}, \dots, a_{i_{k(\alpha)}}; b) \wedge \neg\delta_\alpha(a_{j_1}, \dots, a_{j_{k(\alpha)}}; b),$$

for some $i_1 < \dots < i_{k(\alpha)}$ and $j_1 < \dots < j_{k(\alpha)}$. As α varies in $|T|^+ > |T|$, we can assume $\delta_\alpha = \delta$ for $|T|^+$-many α. Hence, we can construct a sequence $i_1^l < \dots < i_k^l < i_1^{l+1} < \dots < i_k^{l+1} < \dots$ such that $\delta(a_{i_1^l}, \dots, a_{i_k^l}; b)$ holds if and only if l is even. This contradicts the fact that $\phi(x; y)$ is NIP. ■

Now, let $\phi(x; y)$ be a formula, where $y = y_1 \dots y_n$ is an n-tuple. We want to prove that for any indiscernible sequence $(a_i : i < |T|^+)$ and n-tuple $b = b_1 \dots b_n$, the truth value of $\phi(a_i, b)$ is definitely constant. By the claim, there is an $\alpha_1 < |T|^+$ such that the sequence $(a_i : \alpha_1 < i < |T|^+)$ is indiscernible over b_1. This means that the sequence $(a_i b_1 : \alpha_1 < i < |T|^+)$ is indiscernible. Iterating this procedure, we can find a succession $\alpha_1 < \dots < \alpha_n < |T|^+$ such that $(a_i b_1 \dots b_k : \alpha_k < i < |T|^+)$ is indiscernible. In particular, as $(a_i b_1 \dots b_n : \alpha_n < i < |T|^+)$ is indiscernible, the truth value of $\phi(a_i; b)$ is constant for $i > \alpha_n$. □

Using these theorems, we can finally give our first example of a NIP theory.

Example 1.7 (Dense linear orders)**.** Let T be the theory of dense linear orders without extremes. We want to prove that any formula $\phi(x; y) \in L$ is NIP. Using symmetry and Proposition 1.6, we can assume that $|x| = 1$. Moreover, by quantifiers elimination, $\phi(x; y)$ is equivalent to a boolean combination of formulas of the form $x \leq y$. Threfore, by Lemma 1.5, it sufficies to prove that the formula $\psi(x; y) \equiv x \leq y$ is NIP. Suppose by contradiction that there is an indiscernible sequence $(a_i : i \in \omega)$ and an element $b \in \mathcal{U}$ such that $\phi(a_i, b)$ holds if and only if i is even. In particular, this means that $a_0 \leq b$ and $b \leq a_1$. Since $(a_i : i < \omega)$ is indiscernible, this implies $b \leq a_1 \leq a_2$. Therefore, $\phi(a_2, b)$ does not hold. Contradiction.

We now briefly recall the notion of forking.

Definition 1.8. Let $A \subseteq \mathcal{U}$.

(i) A formula $\phi(x; b) \in L(\mathcal{U})$ divides over A if there is an A-indiscernible sequence $(b_i : i < \omega)$ starting with b such that the partial type $\{\phi(x; b_i) : i < \omega\}$ is inconsistent.

(ii) A partial type $\pi(x)$ divides over A if it implies a formula which divides over A;

(iii) A partial type $\pi(x)$ forks over A if it implies a finite disjunction $\bigvee_{i<n} \phi_i(x; b)$ of formulas such that, for each $i < n$, $\phi_i(x; b)$ divides over A.

Clearly, a global type forks if and only if it divides. However, it can happen, even in a NIP theory, that for a partial type forking does not equal dividing. On the other hand, forking and dividing over a model do coincide in NIP theories, although we won't use this fact. The main advantage of forking over dividing is given by the following proposition.

Proposition 1.9. *Let $A \subseteq B$ and $\pi(x)$ be a partial type over B, then $\pi(x)$ does not fork over A if and only if it has an extension to a global complete type which does not divide (equiv. fork) over A.*

Proof. Let $\pi(x)$ be a partial type over A, and let $\Sigma(x)$ be the collection of all formulas $\neg\phi(x; b)$ where $\phi(x; b)$ forks over A. If $\pi(x) \cup \Sigma(x)$ is consistent, then it extend to a global type that does not fork over A. So, suppose $\pi(x) \cup \Sigma(x)$ is not consistent. Then there is a finite subset $\Sigma_0(x) \subseteq \Sigma(x)$ such that $\pi(x) \cup \Sigma_0(x)$ is inconsistent. Thus, $\pi(x)$ implies a disjunction of dividing formulas. $\square$

We conclude this section with a convenient characterization of types non forking over a model M in an NIP theory.

Proposition 1.10. *In a NIP theory, a type p is M-invariant if and only if it does not fork on M.*

Proof. Suppose p is not M-invariant. Then there are $c_0, c_1 \in \mathcal{U}$ with $c_0 \equiv_M c_1$ and $p \vdash \phi(x; c_0) \wedge \neg\phi(x; c_1)$. Let $(c_i c_{i+1} : i < \omega)$ be an M-indiscernible sequence of couples starting with $c_0 c_1$. As the sequence $(c_i : i < \omega)$ is M-indiscernible too, the partial type $\{\phi(x; c_{2i}) \wedge \neg\phi(x; c_{2i+1})\}$ is inconsistent by NIP. In fact, if it were realized by some $a \in \mathcal{U}$, then the truth value of $\models \phi(a, c_i)$, for $i < \omega$, would be alternating. Hence, the formula $\phi(x, c_0) \wedge \phi(x, c_1)$ divides over M.

Conversely, suppose p is M-invariant. Let $p \vdash \phi(x; b)$ and let $(b_i, i < \omega)$ be any indiscernible sequence starting with $b_0 = b$. Then, by invariance, $p \vdash \phi(x; b_i)$ for every $i < \omega$. Hence, the partial type $\{\phi(x; b_i) : i < \omega\}$ is consistent. Notice that this implication does not use NIP. $\square$

1.2 Keisler Measures

In this section we define a Keisler measure and we describe three different ways in which we can codify a measure over a model. Finally, we prove some basic extension results.

Definition 1.11. Let $\mathcal{L}_A$ be the algebra of the A-definable sets. A Keisler measure over A is any finitely additive measure on $\mathcal{L}_A$.

Notice that we can identify $\mathcal{L}_A$ with the clopen sets of $S(A)$. Under this identification, the σ-algebra generated by $\mathcal{L}_A$ coincides with the σ-algebra of Borel sets of $S(A)$. Many nice properties of Keisler measures come from the particular topological structure of the Stone space of types.

Lemma 1.12. *Let $\mathcal{A}$ be a basis of clopen subsets of a compact Hausdorff space S, and let μ be a finitely additive measure over $\mathcal{A}$. Then μ extends uniquely to a regular σ-additive measure over the Borel sets of S.*

Proof. Let $O \subseteq S$ be an open set, define $\mu(O) = \sup\{\mu(D) : D \subseteq O \text{ e } D \in \mathcal{A}\}$. Similarly, for $F \subseteq S$ a closed set, define $\mu(F) = \inf\{\mu(D) : D \subseteq O \text{ e } D \in \mathcal{A}\}$. Notice that, as the space is compact and Hausdorff, for each open set O contained in an closed set F, there is a finite union D of elements from $\mathcal{A}$ such that $F \subseteq D \subseteq O$. This implies that for every closed or open set X, we have

$$\sup\{\mu(F) : F \subseteq X,\, F \text{ closed}\} = \inf\{\mu(O) : X \subseteq O,\, O \text{ open}\}. \tag{Reg}$$

In fact, suppose for example that X is open and let $s = \sup\{\mu(F) : F \subseteq X,\, F \text{ closed}\}$. The right hand side term is then trivially equal to $\mu(X)$, so we just have to prove $\mu(X) = s$. As the sets in $\mathcal{A}$ are closed, by definition $\mu(X) \leq s$. On the other hand, if $F \subseteq X$ is closed, then by the previous remark there is D in A such that $F \subseteq D \subseteq X$, thus $\mu(X) \geq s$

Now, we prove the family of subsets such that (Reg) holds is closed under complement and countable union. In particular, this implies that it is a σ-algebra containing the Borel sets. Complement is clear. Now, let $X = \bigcup_{i<\omega} X_i$ be a countable union such that for every X_i equation (Reg) holds. Then we can choose $F_i \subseteq X_i \subseteq O_i$ such that $\mu(O_i) - \mu(F_i) < \varepsilon 2^{-i}$. Let $O = \bigcup_{i<\omega} O_i$. We can find some N such that $\mu(O) - \mu(\bigcup_{i<N} O_i) < \varepsilon$. Then $F = \bigcup_{i<\omega} F_i$ is a closed set such that $F \subseteq X \subseteq O$ and $\mu(O) - \mu(F) < 3\varepsilon$.

It is now easy to check that μ can be extended to the σ-algebra we just defined, and in particular to the Borel sets. (In fact, this is a simple case of the more general Carathéodory extension theorem). □

Proposition 1.13. *There is a bijective correspondence between Keisler measures on $\mathcal{L}_A$ and regular Borel-measures on $S(A)$.*

Proof. Let μ be a finitely additive measure over $\mathcal{L}_A$. By the previous lemma, it extends to a regular Borel measure over $S(A)$. Conversely, if μ is a regular probability measure on $S_x(A)$, then it defines a Keisler measure by restriction to the algebra of clopen sets. □

Keisler measures and regular Borel measures on the space of types are two equivalent and useful ways to codify the measures of a model. We now introduce a third way that is particularly useful to study the extensions of a measure over a model M to an elementary extension $N \succ M$.

Let M be a structure and let μ be a Keisler measure over M. We encode this data as a multi-sorted structure

$$\tilde{M}_\mu = (M, [0,1], <, +, (f_\phi)_{\phi(x;y) \in L(M)}),$$

where M is equipped with its full structure, $[0,1]$ is the unit interval equipped with the standard order and addition modulo 1, and $f_\phi : M \to [0,1]$ is given by $f_\phi(b) = \mu(\phi(x;b))$. Let now $\tilde{N} = (N, [0,1]^*, \ldots)$ be an elementary extension of $\tilde{M}_\mu$, where $[0,1]^*$ is a non standard expansion of $[0,1]$. We can extend μ to N by letting $\mu(\phi(x;b)) = \mathrm{st}(f_\phi(b))$, where $\mathrm{st} : [0,1]^* \to [0,1]$ is the standard part map. We will often use this construction to extend a measure to a bigger model while preserving some elementary property.

As it is the case for types, it is interesting to study the extensions of a given measure. The following lemma tells us that measures can always be extended.

Lemma 1.14. *Let $\Omega \subseteq \mathcal{L}_A$ be a subalgebra, and let μ_0 be a finitely additive probability measure over Ω. Then μ_0 extends to a Keisler measure over $\mathcal{L}_A$.*

Proof. Because the space of finitely additive measure over $\mathcal{L}_A$ is a closed subspace of the compact space $[0,1]^{\mathcal{L}_A}$, it suffices to show that for any finite subalgebra B of $\mathcal{L}_A$ there is a finitely addictive measure over B that coincides with μ_0 over $B \cap \Omega$. To do this, let $\psi_1, \ldots, \psi_n$ be the atoms of B, and let $\phi_1, \ldots, \phi_m$ be the atoms of the subalgebra $B \cap \Omega$. Then, we have

$$\phi_i = \psi_{k_i(1)} \vee \ldots \vee \psi_{k_i(l_i)},$$

for $1 \leq i \leq m$. Notice that each ψ_k may appear in at most one of the expressions for ψ_i, as all the atoms are disjoint. Define μ on the atoms in such a way that, for each $i \leq m$, $\mu_0(\phi_i) = \mu(\psi_{k_i(1)}) + \ldots + \mu(\psi_{k_i(l_i)})$. This defines a finitely addictive measure over B compatible with μ_0. □

An immediate consequence of the previous lemma is that any measure over a small model can be extended to a global measure, as it is easily seen by letting $A = \mathcal{U}$ and $\Omega = \mathcal{L}_M$. Such an extension is not unique in general. The following lemma gives us necessary and sufficient conditions for the existence of multiple extensions.

Lemma 1.15. *Let $\mu \in \mathfrak{M}_x(M)$ be a measure and let $\phi(x;b) \in L(\mathcal{U})$. Let $r_1 = \sup\{\mu(\psi(x)) : \psi(x) \in L(M), \models \psi(x) \to \phi(x;b)\}$ and $r_2 = \inf\{\mu(\psi(x)) : \psi(x) \in L(M), \models \phi(x;b) \to \psi(x)\}$. Then for any $r_1 \leq r \leq r_2$, there is an extension $\nu \in \mathfrak{M}_x(\mathcal{U})$ of μ such that $\nu(\psi(x;b)) = r$.*

Proof. By Lemma 1.14, it suffices to show that we can extend μ to a finitely additive measure ν on the algebra generated by all the $L(M)$-formulas and $\phi(x;b)$ in such a way that $\nu(\phi(x;b)) = r$. First, we find a measure μ' such that $\mu'(\phi(x;b)) = r_2$. To this end, consider $C = \bigcap_{i<\omega} \theta_i(x)$ where $\theta_i(x) \in L(M)$ is a formula such that $\phi(x;b) \to \theta_i(x)$ and $\lim_{i\to\infty} \mu(\theta_i(x)) = r_2$. Notice that, since C is a countable intersection of definable sets, C is measurable. We want to think of C as measurable approximation from the outside of $\phi(x;b)$.

It is easy to see that to define μ' it suffices to define its value on the sets of the form $E = \phi(x;b) \cap B$ for some $B \in \mathcal{L}_M$. In particular, we put $\mu'(E) = \mu(C \cap B)$. By definition of C it follows that μ' is well defined, that is if $E \in \mathcal{L}_A$ then $\mu'(E) = \mu(E)$. From the fact that μ is a measure, it follows that μ is finitely additive over $\mathcal{L}_M(\phi(x;b))$.

Define a measure μ'' similarly using in place of C the measurable set $A = \bigcup_{i<\omega} \theta_i(x)$ where $\theta_i(x) \in L_M$ is a formula such that $\theta_i(x) \to \phi(x;b)$ and $\lim_{i<\omega} \mu(\theta_i(x)) = r_1$. Clearly $\mu''(\phi(x;b)) = r_1$. To conclude we choose ν to be an appropriate interpolation of μ' and μ''; more precisely we put $\nu = (1-\lambda)\mu' + \lambda\mu''$ where $\lambda = (r - r_1)/(r_2 - r_1)$. □

We now give another useful characterization of those measures that have a unique global extension. As we will see, these measures play a central role, hence we give the following definition.

Definition 1.16. Let $\mu \in \mathfrak{M}_x(M)$. We say that μ is *smooth* if, for every $N \supset M$, μ has a unique extension to an element of $\mathfrak{M}_x(N)$. More generally, if $\mu \in \mathfrak{M}_x(N)$ and $M \subseteq N$, we say that μ is smooth over M if $\mu|_M$ is smooth.

Lemma 1.17. *Let $\mu \in \mathfrak{M}_x(M)$ be a smooth measure. Let $\phi(x,y) \in L$ and $\varepsilon > 0$. Then there are formulas $\theta_i^{0,1}(x) \in L(M)$ and $\phi_i(y) \in L(M)$, for $i = 1, \ldots, n$, such that:*

(i) the formulas $\psi_i(y)$ partition the y-space;

(ii) for all i, if $\models \psi_i(b)$, then $\models \theta_i^0(x) \to \phi(x,b) \to \theta_i^1(x)$;

(iii) for each i, $\mu(\theta_i^1(x)) - \mu(\theta_i^0(x)) < \varepsilon$.

Proof. Fix $b \in \mathcal{U}$. We can find $\theta_b^0(x), \theta_b^1(x) \in L(M)$ such that $\theta_b^0(x) \subseteq \phi(x;b) \subseteq \theta_b^1(x)$ and $\mu(\theta_b^1(x)) - \mu(\theta_b^0(x)) < \varepsilon$. In fact, if we could not, then r_2 and r_1 in the statement of Lemma 1.15 would be different, hence there would be multiple extensions of μ.

Now, let $\psi_b(y) \equiv \forall x(\theta_b^0(x) \to \phi(x;y) \to \theta_b^1(x))$. The collection of sets $\psi_b(y)$ as b varies in $\mathcal{U}$ is an open cover of $S(M)$. In fact, let $q \in S(M)$ and $b \in \mathcal{U}$ such that $b \models q$, then $q \vdash \phi_b(y)$. Hence, by compactness, there are finitely many $b_1, \ldots, b_n \in \mathcal{U}$ such that $\psi_{b_1}, \ldots, \psi_{b_n}$ covers $S(M)$. To conclude, take $\theta_i(x) \equiv \theta_{b_i}(x)$ and $\psi_i(y) \equiv \psi_{b_i}(y)$. □

1.3 Approximation of measures

Until now, we still have not used the fact that the theory is NIP. In this section, we will see one of the main implications of the NIP hypothesis for measures, namely Lemma 1.20 below. From this lemma, we will deduce that any measure can be approximated by a finite sum of measures concentrated over a point. In turn, this approximation will be used extensively in the next section to study commutativity of measures.

We will derive Lemma 1.20 as an easy consequence of the following.

Lemma 1.18. *Let $\mu \in \mathfrak{M}_x(M)$ be a measure and $(b_i : i < \omega)$ an indiscernible sequence in M. Let $\phi(x;y)$ be a formula and $r > 0$ such that $\mu(\phi(x;b_i)) \geq r$ for all $i < \omega$. Then the partial type $\{\phi(x;b_i) : i < \omega\}$ is consistent.*

Proof. First we show that we can assume that the sequence $(b_i : i < \omega)$ is μ-indiscernible, that is $\mu(\phi(x;b_{i_1}) \wedge \ldots \wedge \phi(x;b_{i_n})) = \mu(\phi(x;b_{j_1}) \wedge \ldots \wedge \phi(x;b_{j_n}))$ for any $i_1 < \ldots < i_n < \omega$ and $j_1 < \ldots < j_n < \omega$. To do this, expand M to $\tilde{M}_\mu$. In an elementary extension of $\tilde{M}_\mu$ we can find a $\tilde{M}_\mu$-indiscernible sequence realizing the same EM-type of $(b_i : i < \omega)$, call it $(b'_i : i < \omega)$. Notice that the partial type $\{\phi(x;b_i) : i < \omega\}$ is consistent if and only if $\{\phi(x;b'_i) : i < \omega\}$ is consistent, as the two sequences have the same EM-type over M. Therefore, in the rest of the proof we can assume that $(b_i : i < \omega) = (b'_i : i < \omega)$ and in particular that $(b_i : i < \omega)$ is μ-indiscernible.

Now, suppose by contradiction that $\{\phi(x;b_i)\}_{i<\omega}$ is inconsistent. Then there is a minimal $N > 0$ such that $\mu(\phi(x;b_{i_0}) \wedge \ldots \wedge \phi(x;b_{i_N})) = 0$, and, by μ-indiscernibility, we can assume $\mu(\phi(x;b_0) \wedge \ldots \wedge \phi(x;b_N)) = 0$. Because N is minimal, we have $\mu(\phi(x;b_0) \wedge \ldots \wedge \phi(x;b_{N-1})) = t > 0$. Let $\psi_m(x) = \phi(x;b_{m(N-1)}) \wedge \ldots \wedge \phi(x;b_{m(N-1)+N-2})$. Again by indiscernibility, we have $\mu(\psi_m) = \mu(\psi_0) = t > 0$. Also, because N is minimal, we must have $\mu(\psi_m \wedge \psi'_m) = 0$ if $m \neq m'$. It follows that $\mu(\psi_0 \vee \ldots \vee \psi_m) = mt$, contradicting the fact the space has measure 1. □

From this, we can immediately derive a generalization of Proposition 1.10 to measures.

Corollary 1.19. *Let $\mu \in \mathfrak{M}(\mathcal{U})$ be an M-invariant measure. Then, μ does not fork over M, that is every definable set of positive μ-measure does not fork over M.*

Proof. Let $\phi(x, b) \in L(\mathcal{U})$ be a formula such that $\mu(\phi(x, b)) > \varepsilon$ for some $\varepsilon > 0$, and let $(b_i : i < \omega)$ be an indiscernible sequence over M starting with b. Then, since μ is M-invariant, $\mu(\phi(x, b_i)) = \mu(\phi(x, b_j)) = \varepsilon$. Therefore, by Lemma 1.18, the partial type $\{\phi(x, b_i) : i < \omega\}$ is consistent. Hence, $\phi(x, b)$ does not fork. □

Lemma 1.20. *(T NIP) Let $\mu \in \mathfrak{M}_x(M)$. We cannot find a sequence $(b_i : i < \omega)$ of tuples of M, a formula $\phi(x; y)$, and some $\varepsilon > 0$ such that $\mu(\phi(x; b_i)\Delta\phi(x; b_j)) > \varepsilon$ for all $i, j < \omega$, $i \neq j$.*

Proof. By extracting a subsequence, we may assume that the sequence $(b_i : i < \omega)$ is indiscernible. By NIP the partial type $\{\phi(x, b_{2k})\Delta\phi(x; b_{2k+1})\}_{k<\omega}$ is inconsistent. This contradicts the previous lemma. □

As we will see, smooth measures are easier to approximate. Therefore, we would like to reduce from a general measure to a smooth measure. The following proposition gives us a way to do so.

Proposition 1.21. *Let $\mu \in \mathfrak{M}_x(M)$ be any measure. Then, there is $M \prec N$ and an extension $\mu' \in \mathfrak{M}_x(N)$ of μ which is smooth.*

Proof. Let $\kappa = \max(|T|^+, 2^{\aleph_0})$. We construct a sequence of models $(M_\alpha : \alpha < \kappa)$ and measures $\mu_\alpha \in \mathfrak{M}(M_\alpha)$ such that $\mu_{\alpha+1}$ extends μ_α. At the limit step we take the union.

Now, suppose we have defined M_α and μ_α. If μ_α is not smooth, then there is some $b_\alpha \in \mathcal{U}$, a formula $\phi(x; y) \in L(M_\alpha)$, a real number $\varepsilon_\alpha > 0$ and two measures μ' and μ'' such that $\mu''(\phi_\alpha(x; b_\alpha)) - \mu'(\phi_\alpha(x; b_\alpha)) > 4\varepsilon_\alpha$. This means that for any $\theta(x) \in L(M_\alpha)$, the set $\theta(x)\Delta\phi_\alpha(x; b_\alpha)$ has either μ'-measure or μ''-measure $> 2\varepsilon_\alpha$. It follows that, if we let $\mu_{\alpha+1} = \frac{1}{2}(\mu' + \mu'')$, then we have $\mu_{\alpha+1}(\theta(x)\Delta\phi_\alpha(x; b_\alpha)) > \varepsilon_\alpha$ for any $\theta(x) \in L(M_\alpha)$.

Let $\nu = \bigcup_\alpha \mu_\alpha$. Because α varies in $\kappa > |T|$, by taking a subsequence we can assume that $\phi_\alpha = \phi$ and $\varepsilon_\alpha = \varepsilon$ are constant. Hence, by construction, $\nu(\phi(x; b_\alpha)\Delta\phi(x; b_\beta)) > \varepsilon$ for any $\beta > \alpha$. This contradicts Lemma 1.20. □

We are now ready to state our approximation theorem. Recall that we can identify a type with a 0-1 measure. With this in mind we define the average measure $\mathrm{Av}(p_1, \ldots, p_n)$ of the types $p_1, \ldots, p_n \in S(M)$ as

$$\mathrm{Av}(p_1, \ldots, p_n) = \frac{1}{n}\sum_{i=1}^{n} p_i.$$

Clearly, this is equivalent to say that the measure of a set X is

$$\mathrm{Av}(p_1, \ldots, p_n; X) = \frac{1}{n}\left|\{1 \leq i \leq n : p_i \in X\}\right|.$$

Lastly, if $a_1, \ldots, a_n \in \mathcal{U}$, we shall write $\mathrm{Av}(a_1, \ldots, a_n)$ meaning $\mathrm{Av}(\mathrm{tp}(a_1), \ldots, \mathrm{tp}(a_n))$. With these notations we have:

Lemma 1.22. *Let μ_x be a global measure, smooth over M. Let X be a Borel subset of $S_x(M)$, and $\phi(x;y)$ a formula. Fix $\varepsilon > 0$. Then there are $a_1, \ldots, a_n \in U$ such that for any $b \in U$,*

$$|\mu(X \cap \phi(x;b)) - \mathrm{Av}(a_1, \ldots, a_n; X \cap \phi(x;b))| < \varepsilon$$

and also

$$|\mu(X) - \mathrm{Av}(a_1, \ldots, a_n; X)| < \varepsilon.$$

Proof. Fix $\psi_i(y)$, $\theta_i^0(x)$ and $\theta_i^1(x)$ for $i \leq m$ as in Lemma 1.17. Consider μ as a probability measure over $S(M)$. Using the weak law of big numbers we can find a sequence of types $p_1, \ldots, p_n \in S(M)$ such that for every $i < m$ and $\epsilon \in \{0, 1\}$

$$|\mu(X \cap \theta_i^\epsilon(x)) - \mathrm{Av}(p_1, \ldots, p_n; X \cap \theta_i^\epsilon(x))| < \varepsilon.$$

Choose points $a_1, \ldots, a_n \in \mathcal{U}$ such that $a_i \models p_i$. Notice that

$$\mathrm{Av}(a_1, \ldots, a_n; X \cap \theta_i^\epsilon(x)) = \mathrm{Av}(p_1, \ldots, p_n; X \cap \theta_i^\epsilon(x)).$$

Fix some $b \in \mathcal{U}$ and choose $0 \leq i \leq n$ such that $\theta_i^0(x) \subseteq \phi(x, b) \subseteq \theta_i^1(x)$. From this containment and the fact that $|\mu(\theta_i^1(x)) - \mu(\theta_i^0(x))| < \varepsilon$, we deduce:

$$|\mu(X \cap \phi(x;b)) - \mathrm{Av}(a_1, \ldots, a_n; X \cap \phi(x;b))| \leq 3\varepsilon.$$

□

Proposition 1.23. *Let $\mu \in \mathfrak{M}_x(A)$ be any Keisler measure; let $\phi(x;y) \in L$ be a formula and fix $X_1, \ldots, X_m \subseteq S_x(A)$ Borel subsets. Let $\varepsilon > 0$. Then there are types $p_1, \ldots, p_n \in S_x(A)$ such that, for every $b \in A$ and every $k \leq m$:*

$$|\mu(\phi(x;b) \cap X_k) - \mathrm{Av}(p_1, \ldots, p_n; \phi(x;b) \cap X_k)| < \varepsilon.$$

Proof. Using Proposition 1.21, extend μ to a smooth measure ν over $M \supset A$. The previous lemma tells us that there are some points $a_1, \ldots, a_n \in \mathcal{U}$ such that

$$|\mu(X_k \cap \phi(x;b)) - \mathrm{Av}(a_1, \ldots, a_n; X_k \cap \phi(x;b))| < \varepsilon,$$

for all $b \in \mathcal{U}$ and $1 \leq k \leq m$. To conclude, choose $p_i = \mathrm{tp}(a_i/A)$. □

1.4 Product of measures

In this section we will define, under suitable hypotheses, the product measure $\mu_x \otimes \lambda_y$ on the space of types $S_{xy}(\mathcal{U})$. The definition is more involved than the classic definition of product measure to account for the fact that the space $S_{xy}(\mathcal{U})$ is more complex than the simple product $S_x(\mathcal{U}) \times S_y(\mathcal{U})$. It will be useful to introduce right now some terminology.

Definition 1.24. Let $M \models T$ and let $\mu \in \mathfrak{M}_x(\mathcal{U})$ be a measure. We say that:

- μ is *finitely satisfiable* in M if for every $\phi(x;b) \in L(\mathcal{U})$ such that $\mu(\phi(x;b)) > 0$, there is $a \in M$ such that $\mathcal{U} \models \phi(a;b)$.
- μ is *definable* over M if it is M-invariant and, for every $\phi(x;y) \in L$ and $r \in [0, 1]$, the set $\{q \in S_y(M) : \mu(\phi(x;b)) < r \text{ for any } b \in \mathcal{U}, b \models q\}$ is an open subset of $S_y(M)$.

- μ is *Borel-definable* over M if it is M-invariant and the above set is a Borel set of $S_y(M)$.
- μ is *generically stable* if it is both definable and finitely satisfiable in some small model.

We will need the following technical fact. Notice that, together with Lemma 1.20 this is one of the most important consequences of the NIP hypothesis for the study of measures.

Fact 1.25 ([Sim14, Proposition 7.19]). *(T NIP) Let $\mu \in \mathfrak{M}(\mathcal{U})$ be M-invariant, then μ is Borel-definable over M.*

Sketch of the proof. Using Proposition 1.23 we can reduce to the case where $\mu = p$ is an M-invariant type. Thus, we have to prove that p is Borel-definable. This is done using NIP and the notion of eventual types. □

We are now ready to define the product measure. Let $\mu \in \mathfrak{M}(\mathcal{U})$ be a global M-invariant measure and let $\phi(x, y; b) \in L(N)$ be a formula, where $N \supseteq M$ is a small model. We define a function $F^{\mu}_{\phi,N} : S(N) \to \mathbb{R}$ by letting $F^{\mu}_{\phi,N}(q) = \mu(\phi(x, d; b))$ for some (any) $d \models q$. Since μ is M-invariant, the function is well defined. Moreover, Fact 1.25 tells us that μ is Borel-definable and this translates precisely to the fact that $F^{\mu}_{\phi,N}$ is measurable.

Let now $\lambda \in \mathfrak{M}(\mathcal{U})$ be any measure. We define the product measure $\omega(x, y) = \mu_x \otimes \lambda_y$ by the formula:

$$\mu_x \otimes \lambda_y(\phi(x, y; b)) = \int_{q \in S_y(N)} F^{\mu}_{\phi,N}(q) \, \mathrm{d}\lambda_y|_N,$$

where N is some (any) small model containing Mb. For this to be well defined, we have to check that the value of the integral does not depend on the choice of N. To do end, let $N_2 \supset N_1$ be two models containing Mb, and let $\pi : S(N_2) \to S(N_1)$ be the standard restriction map. Notice that $F^{\mu}_{\phi,N_2} = F^{\mu}_{\phi,N_1} \circ \pi$ and that $\pi_*(\lambda_y|_{N_2}) = \lambda_y|_{N_1}$. We can then conclude using the following measure theoretic identity:

$$\int_{q \in S(N_2)} F^{\mu}_{\phi,N_1}(\pi(q)) \, \mathrm{d}\lambda_y|_{N_2} = \int_{q \in \pi(S(N_2))} F^{\mu}_{\phi,N_1}(q) \, \mathrm{d}\pi_*(\lambda_y|_{N_2}).$$

Lastly, to make these formulas more easily readable, we shall often write $\mu_x(\phi(x, q))$ in place of $F^{\mu}_{\phi,N}(q)$. With this notation we have

$$\mu_x \otimes \lambda_y(\phi(x, y; b)) = \int_{q \in S_y(N)} \mu_x(\phi(x, q)) \, \mathrm{d}\lambda_q.$$

Notice that this formula is reminiscent, perhaps in a somewhat misleading manner, to the classical formula for the product measure.

In general the product of measures is not commutative. However, as we shall see, there are conditions under which two measures commute. In particular, we will prove that generically stable measures always commute with themselves.

Lemma 1.26. *Let $\mu_x, \lambda_y \in \mathfrak{M}(\mathcal{U})$ be two global M-invariant measures. Assume that $\mu_x \otimes p_y = p_y \otimes \mu_x$ for any $p \in S_y(\mathcal{U})$ in the support of λ_y. Then μ_x and λ_y commute.*

Proof. Let $\phi(x,y)$ be a formula in $L(N)$. By the definition of tensor product

$$\mu_x \otimes \lambda_y(\phi(x,y)) = \int_{q\in S_y(N)} F^{\mu}_{\phi,N}(q)\,\mathrm{d}\lambda_y.$$

As $F^{\mu}_{\phi,N}$ is measurable and bounded, we can find some Borel sets $X_1,\dots,X_n$ that partition $S(N)$ and such that, for each $1\le i\le n$, there are constants $r_i,t_i\in[0,1]$ such that $r_i < F^{\mu}_{\phi,N}|_{X_i} < t_i$ and $\sum_{i<k}\lambda(X_i)(t_i-r_i)<\varepsilon$. Clearly, we have

$$\sum_{i<k}\lambda(X_i)r_i < \int_{q\in S_y(N)} F^{\mu}_{\phi,N}(q)\,\mathrm{d}\lambda_y < \sum_{i<k}\lambda(X_i)t_i.$$

Now, let $\tilde\lambda = \frac{1}{n}\sum_{i\le n} q_i$ be an approximation of λ with respect to the sets $X_1,\dots,X_k$ and to the formula $\phi(x,y)$, as in Proposition 1.23. By definition of product

$$\mu_x \otimes \tilde\lambda_y(\phi(x,y)) = \int_{q\in S_y(N)} F^{\mu}_{\phi,N}(q)\,\mathrm{d}\tilde\lambda_y.$$

Using the same estimate as before, we have

$$\sum_{i<k}\tilde\lambda(X_i)r_i < \int_{q\in S_y(N)} F^{\mu}_{\phi,N}(q)\,\mathrm{d}\tilde\lambda_y < \sum_{i<k}\tilde\lambda(X_i)t_i.$$

The previous inequalities, together with $|\tilde\lambda(X_i)-\lambda(X_i)|<\varepsilon$, give us

$$|\mu_x\otimes\tilde\lambda_y(\phi(x,y)) - \mu_x\otimes\lambda_y(\phi(x,y))| < \varepsilon.$$

Now, we make the measures commute, and we derive the estimate

$$|\lambda_y\otimes\mu_x - \tilde\lambda_y\otimes\mu_x| \le \int_{q\in S_x(N)} |\lambda(\phi(q,y)) - \tilde\lambda(\phi(q,y))|\,\mathrm{d}\mu_x < \varepsilon.$$

As $\mu_x\otimes\tilde\lambda_y = \tilde\lambda_y\otimes\mu_x$ and $\varepsilon>0$ is arbitrary, putting the two estimates together we finally deduce $\mu_x\otimes\lambda_y = \lambda_y\otimes\mu_x$. □

Lemma 1.27. *Let $\mu_x,\lambda_y\in\mathfrak{M}(\mathcal{U})$ be global measures. Suppose further that there is a model M such that μ_x is definable in M and that λ_y is finitely satisfiable in M. Then, $\mu_x\otimes\lambda_y = \lambda_y\otimes\mu_x$.*

Proof. Since types in the support of a finitely satisfiable formula are finitely satisfiable, using the previous lemma we can restrict to the case where $\lambda = q$ is a type. Suppose $\mu_x\otimes q_y \ne q_y\otimes\mu_x$. This means that we can find a formula $\phi(x,y)$, a $r\in\mathbb{R}$ and a $\varepsilon>0$ such that $\mu_x\otimes q_y(\phi(x,y)) < r-2\varepsilon$ and $q_y\otimes\mu_x(\phi(x,y)) > r+2\varepsilon$. Unravelling the definitions, we have

$$\mu_x\otimes q_y(\phi(x,y)) = \mu(\phi(x,b))$$

for some $b\models q$. Conversely,

$$q_y\otimes\mu_x(\phi(x,y)) = \int_{p\in S_x(M)} q(\phi(p,y))\,\mathrm{d}\mu = \mu(X)$$

where X is the Borel set defined by $\mathrm{tp}(a/M)\in X \Leftrightarrow q\vdash\phi(a,y)$ (recall that q is Borel-definabile by Fact 1.25). In particular, we obtain that $\mu(\phi(x,b)) < r-2\varepsilon$ and $\mu(X) > r+2\varepsilon$.

Now, let $\tilde{\mu} = \frac{1}{n}\sum_{i<n} p_i$, with $p_i \in S_x(M)$, be an approximation of μ with respect to X and $\phi(x,y)$, and fix $a_1, \dots, a_n \in \mathcal{U}$ such that $a_i \models p_i$.

By definability of μ_x, there is a formula $\psi(y) \in L(M)$ such that $q \vdash \psi(y)$ and such that $\mu(\phi(x,b)) < r - \varepsilon$ if and only if $b \in \psi(\mathcal{U})$. As q is finitely satisfiable, we can find some $b_0 \in \psi(M)$ such that $\models \phi(a_i, b_0) \Leftrightarrow q \vdash \phi(a_i, y)$ for all $0 \leq i \leq n$. By explicit computation, we have

$$\tilde{\mu}(\phi(x,b_0)) = \frac{1}{n}\left|\{i : \models \phi(a_i,b_0)\}\right| = \frac{1}{n}\left|\{i : a_i \in X\}\right| = \tilde{\mu}(X) > r,$$

where in the last inequality we used that $\mu(X) > r+\varepsilon$ and that $\tilde{\mu}$ approximates μ. However, we also have $\tilde{\mu}(\phi(x,b_0)) < r$, as $\mu(\phi(x,b_0)) < r-\varepsilon$ and $\tilde{\mu}$ approximates μ. Contradiction. □

Corollary 1.28. *If μ is a generically stable measure, then $\mu_x \otimes \mu_y = \mu_y \otimes \mu_x$.*

The converse is also true, i.e. a global M-invariant measure is generically stable if and only if it commutes with itself [Sim14, 7.29]. However, we won't need it.

1.5 Border of a formula

In this short section, we introduce the fundamental concept of the border of a formula. This will give us a useful way to characterize smooth measures.

Definition 1.29. Let $M \models T$ and $f : M \to C$ be a function to a set C. For $\phi(x,y) \in L(M)$ and $b \in \mathcal{U}$, we define the f-border of $\phi(x,b)$ as $\partial_f^M \phi(x,b) = f(\phi(M,b)) \cap f(\neg\phi(M,b))$

The most important notion of border is the $\mathrm{tp}(\cdot/M)$-border, corresponding to the function $\mathrm{tp}(\cdot/M) : M \to S(M)$. In the following we will refer to it simply as the border of a formula, and we will write it as ∂^M. It is easy to check that an equivalent way to define ∂^M is

$$\partial^M \phi(x,b) = \{p \in S_x(M) : \text{ there are } a, a' \models p \text{ s.t. } \models \phi(a,b) \wedge \neg\phi(a',b)\}.$$

The border of a formula gives us the following nice characterisation of smooth measures.

Lemma 1.30. *Let $\mu \in \mathfrak{M}_x(M)$. Then μ is smooth if and only if for any $\phi(x,y) \in L(M)$ and $b \in \mathcal{U}$ the border $\partial^M \phi(x,b)$ has μ-measure zero.*

Proof. ($\Leftarrow$) Let $O \subseteq S_x(M)$ be the set of types such that $p \vdash \phi(x,b)$, let $F = \partial^M \phi(x,b)$ and let ν be an extension of μ. Notice that O is the union of the formulas $\psi(x) \in L(M)$ such that $\mathcal{U} \models \psi(x) \to \phi(x,b)$ and that $O \cup F$ is the intersection of the formulas $\psi(x) \in L(M)$ such that $\phi(x) \to \psi(x)$. Hence, O is open and F is closed and both are measurable. Now, we clearly have $\mu(O) < \nu(\phi(x,b)) < \mu(O) + \mu(F)$. If $\mu(F) = 0$ for every ϕ and b, then it follows that $\nu(\phi(x,b)) = \mu(O)$ Hence, the measure ν is completely determined by μ, and so μ is smooth, since it has at most one extension.

($\Rightarrow$) Use the same notation as before. Recall that O is the union of all definable sets contained in $\phi(x,b)$ and $O \cup F$ is the intersection of all definable sets containing $\phi(x,b)$. Suppose that $\mu(F) > 0$. Then, $\mu(O) < \mu(O \cup F)$. We can then apply Lemma 1.15 to find multiple extensions of μ. □

Chapter 2

NIP groups

In this chapter we will talk about groups definable in NIP theories. In particular, we will establish the existence of the connected component G^{00} (Section 2.1) and describe the logic topology, that makes G/G^{00} a topological compact group (Section 2.2). We will then study the properties of invariant measures (Section 2.3). This will lead us to consider the notion of fsg groups, or measure stable groups, as they have more recently been called to emphasize that they are the natural generalization of stable groups in the NIP setting.

The rest of the chapter (Sections 2.4 and 2.5) is dedicated to the proof of the compact domination property for fsg groups having a smooth invariant measure. It should be noted that in an o-minimal theory this last condition is always satisfied (cf. Theorem 4.48), so this section effectively prove compact domination for fsg groups in o-minimal theories.

Perhaps the most important idea to prove the compact domination property is the ability to find elements that are in some sense "random" with respect to the unique global invariant measure of the group. The delicate definition of such elements and the proof of their properties are carried out in Section 2.5.

Lastly, we sould remark that most of the results and proofs of this chapter apply verbatim to type-definable groups. However, for simplicity, we shall limit ourselves to the definable case.

2.1 Connected components

In this section we prove the existence of the connected components of a group G definable in a NIP theory. Recall that we say that a group $G \subseteq \mathcal{U}$ is definable if G is definable as a set and the group operation $\cdot : G \times G \to G$ is definable.

Let A be a small set of parameters. We define G^0_A to be the intersection of all A-definable subgroups of G of finite index. Notice that G^0_A itself can have infinite index in G, however the index of G^0_A is always bounded. It could happen that G^0_A does not actually depends on A, and so it is always equal to $G^0_\emptyset$. If this is the case we put $G^0 = G^0_\emptyset$ and we say that the *connected componet* G^0 exists. We will often use the following criterion for the existence of G^0.

Lemma 2.1. *G^0 exists if and only if the intersection of all subgroups of G of finite index has bounded index.*

Proof. Let $\tilde{G}^0$ be the intersection of all subgroups of finite index. Notice that G^0 exists if and only if $G^0_\emptyset = \tilde{G}^0$. Now, suppose that G^0 exists, and hence that $G^0_\emptyset = \tilde{G}^0$. Then, since $G^0_\emptyset$ has bounded index, the same must hold for $\tilde{G}^0$.

Conversely, suppose that $\tilde{G}^0$ has bounded index. Then, since it is $\emptyset$-invariant, it must be type-definable over any small model (cf. Proposition 2.10). □

In NIP theories G^0 always exists. To prove this, we will use the following theorem, that is also important on its own. Recall that we say that a family of subgroup H_a is uniformly definable if there is a formula $\phi(x, y)$ such that H_a is defined by $\phi(x, a)$.

Theorem 2.2 (Baldwin-Saxl)**.** *Let G be a group definable in an NIP theory T and let H_a be a uniformly definable family of subgroups of G. Then, there is an integer N such that for any* finite *intersection $\bigcap_{a\in A} H_a$, there is a subset $A_0 \subseteq A$ of size N with $\bigcap_{a\in A} H_a = \bigcap_{a\in A_0} H_a$.*

Proof. Since H_a is uniformly definable, there is a formula $\phi(x, y)$ such that $H_a = \phi(x, a)$. Without loss of generality, assume that $\phi(x, a')$ defines a group for any $a' \in \mathcal{U}$.
Claim: Suppose the theorem does not hold for N. Then the formula $\phi^{opp}(x, y)$ must have VC-dimension at least N.
Proof: Suppose A is a counterexample to the fact that the theorem holds for N. Let $A' = \{a_0, \ldots, a_{N'}\}$ be the smallest subset of A such that $\bigcap_{a\in A'} H_a = \bigcap_{a\in A} H_a$. Since the theorem does not hold for N and A, we must have $N' \geq N$. Notice that my minimality of A', for every $k \leq N'$, we have $\bigcap_{a\in A'\setminus\{a_k\}} H_a \neq \bigcap_{a\in A'} H_a$. Let $K_k = \bigcap_{a\in A'\setminus\{a_k\}} H_a$ and $K = \bigcap_{a\in A'} H_a$. For every $k \leq N'$, pick a point $c_k \in K_k \setminus K$, and for any subset $B \subseteq \{0, \ldots, N' + 1\}$, define $c_B = \prod_{k\in B} c_k$. Then, we have

$$c_B \in H_{a_k} \iff k \notin B.$$

This shows that the formula $\phi^{opp}(y; x)$ has VC-dimension at least $N' \geq N$, as we claimed. ■

Now, since the theory is NIP, the VC-dimension of $\phi^{opp}(x; y)$ must be finite. Hence, using the claim, we can conclude that there is a maximal N such that the theorem does not hold for N. □

Remark 2.3. The conclusion of Theorem 2.2 need not apply to infinite intesections.

Lemma 2.4. *Let $(H_i : i \in I)$ be a family of subgroups of a group G and suppose that there is a $K < \omega$ such that every finite intersection of the H_i has index less than K. Then $\tilde{H} = \bigcap_{i\in I} H_i$ has index less than K.*

Proof. Let $i_1, \ldots, i_n \in I$ be such that $H' = H_{i_1} \cap \ldots \cap H_{i_n}$ realizes the maximum index realized by finite intersection. Then for any other H_i in the family, we must have $H' \cap H_i = H'$. Hence, $H' = \tilde{H}$. □

Theorem 2.5. *(T NIP) Let G be a definable group. Then, G^0 exists.*

Proof. By Lemma 2.4, it sufficies to show that the intersection of all subgroups of finite index has bounded index. First, we will use Theorem 2.2 to show that the intersection of all the subgroups defined by a fixed formula ϕ, as the parameters vary in $\mathcal{U}$, has bounded index. Then, we will conclude easily using the fact that the number of formulas is bounded.

Let $\phi(x,y) \in L$ be a formula and consider any family of subgroup $(H_i : i \in I)$ such that $H_i = \phi(x, b_i)$ and, for each i, $[G : H_i] \leq k$. By Theorem 2.2, we can rewrite any finite intersection of such subgroups as the intersection of at most N_ϕ subgroups, for some N_ϕ depending only on the theory. Therefore, the index of any finite intersection of subgroups in H_i has index $\leq kN_\phi$. By Lemma 2.4, the same also holds for infinite intersections.

Let G_ϕ, k be the intersection of all subgroups defined by ϕ and with index $\leq k$. By what we just said, it has finite index. It follows that $G_\phi = \bigcap_k G_{\phi,k}$ too has bounded index. Now, notice that $G^0 = \bigcap_\phi G_\phi$. Because there are boundedly many formulas, and the intesection of boundedly many subgrops of bounded index has bounded index, we deduce that G^0 too has bounded index. □

We define the connected component G^{00} similarly to G^0. Let A be a small set of parameters, we define G^{00}_A as the intersection of all type-definable over A subgroups of bounded index. Clearly, G^{00}_A itself is a type-definable over A subgroup of bounded index, hence it is the smallest such subgroup.

If G^{00}_A does not depends on A, we put $G^{00} = G^{00}_\emptyset$ and we say that the *connected componet* G^{00} exists. Notice that,when it exists, G^{00} is the smallest type-definable subgroup of bounded index. Reasoning as in Lemma 2.1 we have:

Lemma 2.6. *G^{00} exists if and only if the intersection of all type-definable subgroups of bounded index has itself bounded index.*

In NIP theories G^{00} always exists. To prove this we will make use of the following useful lemma from [BOPP05, Remark 1.4].

Lemma 2.7. *Suppose H is a type-definable subgroup of a definable group G. Then there are type-definable subgroups H_i of G for $i \in I$ (with $|I| < \kappa$) such that each H_i is defined by at most a countable set of formulas, and with $H = \bigcap_{i \in I} H_i$.*

Proof. Suppose that H is defined by $\{\phi_j(x) : j \in J\}$, where $|J| < \kappa$. We may assume this set of formulas to be closed under intersection and and that if $H \models \forall x(\phi_j(x) \to \phi_j(x^{-1})$. Now, fix $j_0 \in J$. By compactness, we can find $j_1 \in J$ such that $H \models \forall x, y(\phi_{j_1}(x) \wedge \phi_{j_1}(y) \to \phi_{j_0}(x \cdot y))$. Proceed this way to find a sequence $j_0, j_1, j_2, \ldots \in J$. Then the set of formulas $\{\phi_{j_i}(x) : i < \omega\}$ defines a subgroup. To conclude, repeat this costruction starting with any $j \in J$. □

Theorem 2.8. *(T NIP) Let G be a definable group. Then, G^{00} exists.*

Proof. Notice that the intesection of boundedly many type-definable subgroups of bounded index has bounded index. Hence, by contradiction, if G^{00} does not exist, by Lemma 2.6, then there must exist an arbitrarily large collection $\mathcal{H}$ of pairwise distinct type-definable subgroups of bounded index. By Lemma 2.7 we can assume the subgroups in $\mathcal{H}$ to be defined by at most countably many formulas.

Notice that, since the number of possible formulas is $|T|$, there are less than $|T|^{\aleph_0}$ possible combination of formulas defining each group in $\mathcal{H}$ (not counting parameters). Therefore, arbitrarily many H_i are defined by the same formulas: that is, $H_i = \Phi(x, \bar{b}_i)$ where Φ is a countable collection of formulas and each

$\bar{b}_i$ is a countable sequence of parameters. To simplify things, substitute the sequence $(\bar{b}_i : i \in I)$ with an indiscernible sequence having the same EM-type and indicized on ω. Notice that this does not change the fact that the subgroups $H_i = \phi(x, \bar{b}_i)$ are pairwise distinct.
Claim: H_i does not contain the intersection $\bigcap_{j \neq i < \omega} H_j$.
Proof: Assume that H_i contains the intersection. We stretch the sequence $(\bar{b}_j : j < \omega)$ inserting in place of b_i a very long sequence $(\bar{b}'_l : l < \kappa')$ such that the whole sequence is still indiscernible. Let $H'_l = \phi(x, \bar{b}'_l)$. By construction, each H'_l contains $\bigcap_{j \neq i} H_j$, and they are all pairwise distinct. However, $\bigcap_{j \neq i} H_j$ has bounded index, so there can be only boundedly many subgroups over it. Take κ' large enough to get a contradiction. ■

Now, we find a formula $\theta(x, y)$ that has IP. By the claim, for each $i < \omega$ we can find an $a_i \in \bigcap_{j \neq i} H_j \setminus H_i$. Moreover, we can choose it in such way that the sequence $(a_i, \bar{b}_i : i < \omega)$ is indiscernible. By construction both $\neg\Phi(a_0, \bar{b}_0)$ and $\Phi(a_0, \bar{b}_1)$ hold. By compactness, we can find a formula $\phi \in \Phi$ such that $\neg\phi(a_0, \bar{b}_0)$ and $\phi(a_0, \bar{b}_1)$ hold.

As the sequence $(a_i, \bar{b}_i : i < \omega)$ is indiscernible, we have $\models \neg\phi(a_i, \bar{b}_i) \wedge \phi(a_i, \bar{b}_j)$, for $i \neq j < \omega$. Using the fact that Φ defines a subgroup and that Φ implies ϕ, we see that $\models \bigwedge_{i=1,2,3} \Phi(x_i, y) \to \phi(x_1 \cdot x_2 \cdot x_3, y)$. By compactness, this means that there is a formula $\theta(x, \bar{y})$ implied by $\Phi(x, \bar{y})$ such that $\models \bigwedge_{i=1,2,3} \theta(x_i, y) \to \phi(x_1 \cdot x_2 \cdot x_3, y)$. We claim that θ has IP.

Let $I = \{a_{i_1}, \ldots, a_{i_n}\} \subset \omega$ and $a_I = a_{i_1} \cdot \ldots \cdot a_{i_n}$. We prove that $\theta(a_I, b_i)$ holds if and only if $i \notin I$, meaning θ has IP. Let's assume first $i \notin I$. Then by construction $\Phi(a_j, b_i)$ holds for every $j \in I$, hence, as Φ is a group, $\Phi(a_I, b_i)$ holds too, and therefore $\theta(a_I, b_j)$ holds, as it is implied by Φ. Conversely, if $i \in I$, then there are $c_0, c_1 \in H_i$ such that $a_i = c_0 \cdot a_I \cdot c_1$, hence $\neg\phi(c_0 \cdot a_I \cdot c_1, \bar{b}_i)$ holds. As $\theta(c_0, \bar{b}_i)$ and $\theta(c_1, \bar{b}_i)$ both holds, by construction we must have $\neg\theta(a_I, \bar{b}_i)$. □

Example 2.9 ([HPP08, discussion after Proposition 6.1])**.** Let us fix a compact Hausdorff group $\langle G, \cdot, \ldots \rangle$ equipped with additional first order structure. We use the term G to also denote this structure. Let us assume that

(i) $\mathrm{Th}(G)$ has the NIP

(ii) any definable subset of G is Haar measurable (with respect to the unique normalized Haar measure on G)

(iii) there is a neighbourhood basis of the identity of G consisting of definable sets, say U_i for $i \in I$.

Let G^* be a saturated elementary extension of G. Let $\bigcap_{i \in I} U_i^*$ be the group of "infinitesimals" of G^*, denoted by $\inf(G^*)$. Then $G^{00} = \inf(G^*)$ and the quotient G/G^{00}, with the logic topology, is precisely G.

Lastly, we say something about G^∞. Let A be a small set of parameters, we define G_A^∞ as the intersection of all bounded index type-definable subgroups invariant over A. As before, if G_A^∞ does not depend on A, we put $G^\infty = G_\emptyset^\infty$ and we say that G^∞ exists. In NIP theories G^∞ always exists [Sim14, Theorem 8.7], but we won't use this fact. It is interesting to notice that G^∞ is generated by the elements of the form $\{a \cdot b^{-1} : \mathrm{tp}(a/M) = \mathrm{tp}(b/M)\}$. One can easily verify that, if G^∞ exists, then $\{a \cdot b^{-1} : \mathrm{tp}(a/M) = \mathrm{tp}(b/M)\} \subseteq G^\infty$. For the other inclusion refer to [Sim14].

2.2 Logic topology

In this section we present an important way to associate to any bounded type-definable equivalence relationship a compact space that depends only on the theory, and not on any particular model.

Recall that we say that a type-definable equivalence relationship is bounded if it has only boundedly many equivalence classes.

Proposition 2.10. *Let E be an A-invariant type-definable equivalence relation. Then, E is bounded if and only if for any small model $M \supseteq A$ we have that $a \equiv_M b$ also implies aEb.*

Proof. If $a \equiv_M b$ implies aEb for some small model M, then there are at most $|S(M)| \leq |T|^{|M|}$ equivalence classes for E.

Conversely, suppose that E is bounded. First, we prove that this implies that for any A-indiscernible sequence $(a_i : i < \omega)$ and $i, j < \omega$, we have $a_i E a_j$. In fact, suppose that for each $i \neq j < \omega$ we have $\neg a_i E a_j$. Then, we can also find an indiscernible sequence $(a_i : i < |\mathcal{U}|)$ with the same property, contradicting the fact that E is bounded.

Now, suppose that $a \equiv_M b$ and let p be the coheir of $\mathrm{tp}(a/M)$. Let $(a_i : i < \omega)$ be an indiscernible sequence of p over Mab. Then, both $(a) + (a_i : i < \omega)$ and $(b) + (a_i : i < \omega)$ are indiscernible sequences. Then, by the previous paragraph, we have aEa_0Eb. □

Let E be a bounded $\emptyset$-type-definable equivalence relation. We define the *logic topology* over the quotient X/E by saying that a subset $F \subseteq X/E$ is closed if and only if $\pi^{-1}(F)$ is type-definable over some (any) model M.

Proposition 2.11. *The space X/E equipped with the logic topology is a compact Hausdorff space.*

Proof. Let M be any small model, by Proposition 2.10 we have a well-defined surjective map $f : S(M) \to X/E$. By definition of logic topology this map is continuous so, being the image of a compact space by a continuous map, X/E is compact. We now show that it is Hausdorff. Let $a, b \in \mathcal{U}$ be such that $\neg aEb$. Then we have $\models aEx \wedge xEb \to \neg xEy$. By compactness, we can find a formula $\phi(x, y) \in L(\mathcal{U})$ implied by xEy and such that $\models \phi(x, A) \wedge \phi(y, b) \to \neg xEy$. Let $O_a = \{x \in \mathcal{U}/E : \pi^{-1}(x) \subseteq \phi(\mathcal{U}, a)\}$ and $O_b = \{y \in \mathcal{U}/E : \pi^{-1}(y) \subseteq \phi(\mathcal{U}, b)\}$. Then O_a and O_b are two disjoint open neighbourhood of $\pi(a)$ and $\pi(b)$ respectively. □

By definition of G^{00}, the equivalence relation on G defined by $a \sim b \Leftrightarrow aG^{00} = bG^{00}$ is bounded and $\emptyset$-type-definable. Hence, it makes sense to consider the logic topology on $G(\mathcal{U})/G^{00}$.

Proposition 2.12. *The group G/G^{00} equipped with the logic topology is a compact topological group.*

Proof. One has to prove that the group operation and the inverse map are continuous. This is an easy check. □

2.3 Invariant measures

By similarity with the theory of stable groups, NIP groups have initially been classified basing on the existence of particular types, in particular f-generic types and fsg types. In this section we establish a connection between the existence of such types and the existence of global invariant measures with particular properties.

Definition 2.13. A global type $p \in S_G(\mathcal{U})$ is left *f-generic* over A if no left translate of p forks over A.

Lemma 2.14. *Assume that G admits a global f-generic type p over some small model M. Then,* $\mathrm{Stab}_l(p) = G^{00}$.

Proof. We use from the following characterization of the stabilizer of p.
Claim: $\mathrm{Stab}_l(p) = \{g_1 \cdot g_2^{-1} : g_1 \equiv_M g_2\}$
Proof: Assume that $g_1, g_2 \in G$ have the same type over M, we show that $g_1 \cdot p = g_2 \cdot p$. Let $f \in \mathrm{Aut}(\mathcal{U}/M)$ be an automorphism that maps g_1 to g_2. For any $\phi(x; b) \in L(\mathcal{U})$, we have

$$g_1 \cdot p \vdash \phi(x; b) \iff f(g_1) \cdot p \vdash \phi(x; f(b)) \iff g_2 \cdot p \vdash \phi(x; b),$$

where the last equivalence uses the fact that by Proposition 1.10 the type $g_2 \cdot p$ is M-invariant. It follows that $g_1 \cdot g_2^{-1} \in \mathrm{Stab}_l(p)$.

Conversely, let $h \in \mathrm{Stab}_l(p)$ and let $a \models p|_{Mh}$. Then $h \cdot a \equiv_M a$ and $h = (h \cdot a) \cdot a^{-1}$. ■

By the claim, $\mathrm{Stab}_l(p) = \{g_1 \cdot g_2^{-1} : g_1 \equiv_M g_2\}$. Clearly, the elements of the form $g_1 \cdot g_2^{-1}$, with $g_1 \equiv_M g_2$, are contained in G^{00}, since if $g_1 \equiv_M g_2$, then by Proposition 2.10 g_1 and g_2 are in the same coset of any bounded type-definable over M subgroup. Hence, $\mathrm{Stab}_l(p) \subseteq G^{00}$. On the other hand, from that expression we see that $\mathrm{Stab}_l(p)$ is a type-definable subgroup of bounded index, hence $G^{00} \subseteq \mathrm{Stab}_l(p)$. Therefore, $G^{00} = \mathrm{Stab}_l(p)$. □

Remark 2.15. From the previous proof we can also deduce that, if G admits an f-generic type p, then $G^{\infty} = G^{00} = \mathrm{Stab}_l(p)$. In fact, by the given characterisation of $\mathrm{Stab}_l(p)$ we have $\mathrm{Stab}_l(p) \subseteq G^{\infty} \subseteq G^{00} \subseteq \mathrm{Stab}_l(p)$.

We now present an useful method to construct a measure by averaging an f-generic type.

Lemma 2.16. *Let p be a f-generic type. Define a global measure μ_p by letting $\mu_p(\phi) = h(\{\bar{g} \in G/G^{00} : \bar{g} \cdot p \vdash \phi\})$, where h is the Haar measure of G/G^{00}. Then, the measure μ_p is well-defined and left invariant.*

Proof. Well-definition comes from Lemma 2.14, while the left invariance follows from the left invariance of the Haar measure. □

We call a group with a global left invariant measure a *definably amenable group*. By the previous lemma, the existence of a f-generic type implies defianable amenability. It turns out that the converse also holds. To prove it, we will need the following important lemma.

Lemma 2.17. *Let G be a $\emptyset$-definable group and M a small model. Let λ_0 be a left invariant measure over $G(M)$. Then, λ_0 admits a global extension η which is left invariant and definable.*

Proof. First we prove that we can find a model $N \succ M$ and an extension λ_1 of λ_0 to N which is left invariant and such that λ_1 admits a unique left invariant extension to any bigger model. To do this we proceed exactly as in the proof of Proposition 1.21: we build a succession of models M_α and measure μ_α. If μ_α admits two different left invariant extension, we take $\mu_{\alpha+1}$ to be their average. We also need to prove that a left invariant measure can always be extended while staying left invariant measure. To see this, use the construction after Proposition 1.13 and notice that being a left invariant measure is a first order property in the extended language.

Now that we have obtained λ_1, let η be the unique left invariant global extension. We want to prove that η is definable over N. Fix a $q \in S(N)$ and let $r = \eta(\phi(x, q))$. It suffices to prove that there is an open neighbourhood $\psi(y)$ of q such that, for any $a' \models \psi(y)$, we have $|\eta(x, a') - r| < \varepsilon$. To do this, consider the model $\tilde{N}_{\lambda_1} = (N, [0, 1], f_\phi)_\phi$ encoding the measure. As λ_1 has a unique extension to any bigger model, there can be no elementary extension of $\tilde{N}_{\lambda_1}$ containing a that satisfy $|\eta(\phi(x; a)) - r| > \varepsilon$. So $\mathrm{tp}(a/\tilde{N}_{\lambda_1}) \models |\eta(\phi(x; a)) - r| > \varepsilon$. By compactness, there is some formula $\psi(y) \in tp(a/N)$ such that any $a' \models \psi(y)$ has the same property, as we wanted. □

Using the previous proposition, we can give a characterization of definably amenable groups in term of the existence of f-generic types.

Proposition 2.18. *Let M be a model over which G is defined. Then G admits a global f-generic type over M if and only if G is definably amenable.*

Proof. ($\Rightarrow$) This is just Lemma 2.16.

($\Leftarrow$) Suppose that G is definably amenable. Then, by Lemma 2.17, we can find a definable global invariant measure μ. In particular, μ is invariant over some small model M. Hence, μ does not fork over M. It follows that any type p in the support of μ is also non-forking and, as μ is translation invariant, so is any translate gp of p. Therefore, p is f-generic. □

We now turn our attention to fsg types. As we shall see, they are related with the existence of finitely satisfiable global invariant measures.

Definition 2.19. Let G be a $\emptyset$-definable group. We say that a global type p is an *fsg type* if there is a small model M_0 such that every left translate of p is finitely satisfable in M_0. We say that G is an fsg group if it has an fsg type.

Notice that, since finitely satisfable types are invariant, and hence do not fork, an fsg type is automatically an f-generic type.

Proposition 2.20. *Let G be a definable group. Then G is an fsg group if and only if it admits a global invariant finitely satisfiable measure.*

Proof. ($\Rightarrow$) Let p be an fsg type and let $\mu = \mu_p$ be as in Lemma 2.16. We claim that μ is finitely satisfiable. In fact, suppose that $\mu(X) > 0$ for some definable set X. Then, by definition of μ, there must be some transalate $g \cdot p$ of p such that $X \in g \cdot p$. Since $g \cdot p$ is finitely satisfiable, we can conclude.

($\Leftarrow$) Let μ be a global invariant finitely satisfiable measure. It is easy to verify that any type in the support of μ is an fsg type. □

We now improve this results by showing that an fsg group admits a unique invariant measure, and that this measure is also generically stable, i.e. it is both finitely satisfiable and definable.

Proposition 2.21. *Let G be a definable fsg group, then G admits a unique global left invariant measure μ. Moreover μ is generic, it is the unique right invariant measure and is generically stable.*

Proof. Let p be a fsg type and let $\mu = \mu_p$ where μ_p is the global left invariant measure defined in Lemma 2.16. Recall that, by Proposition 2.20, μ is finitely satisfiable. To show that μ is generically stable and unique we use the following claim.
Claim: Let μ be a global left invariant measure on G finitely satisfiable over M and let λ be a global right invariant measure on G definable over M. Then, $\mu = \lambda$.
Proof: The proof uses the fact that, by Lemma 1.27, μ_x and λ_y commute. Let $\phi(z) \in L(N)$ be a formula. Then, $\mu_x \otimes \lambda_y(\phi(y \cdot x)) = \lambda_y \otimes \mu_x(\phi(y \cdot x))$. We can compute the left hand side using the definition of product measure:

$$\begin{aligned}\mu_x \otimes \lambda_y(\phi(y \cdot x)) &= \int_{q \in S(N')} \mu_x(\phi(q \cdot x)) d\lambda_y = \int_{q \in S(N')} \mu_x(\phi(x)) d\lambda_y \\ &= \mu(\phi(x)).\end{aligned}$$

In the second equality we used that μ is left invariant. Similarly, using this time that λ is right invariant, we have $\lambda_y \otimes \mu_x(\phi(y \cdot x)) = \lambda(\phi(y))$. Therefore, $\mu = \lambda$. ■

Now, by Lemma 2.17 there exists a definable global right invariant measure λ. By the claim we have $\mu = \lambda$, therefore μ is definable and right invariant in addiction to being finitely satisfiable. Hence, it is generically stable. We have also proved that μ is the unique definable right invariant measure on G. As μ is right invariant, we can do the same reasoning as before with λ left invariant, and conclude that μ is also the unique definable global left invariant measure of G.

Lastly, we prove that for any left invariant global measure λ, definable or not, we have $\mu = \lambda$. Let M be a small model and $\phi \in L(M)$ be a formula over M. Consider the restriction of λ to M. By Lemma 2.17 it extends to a definable global left invariant measure λ'. Using the claim, we have $\mu(\phi) = \lambda'(\phi)$ and, as λ and λ' have the same restriction to M, we also have $\lambda'(\phi) = \lambda(\phi)$, so $\mu(\phi) = \lambda(\phi)$. As this is true for any small model M and any formula ϕ over M, we can conclude $\mu = \lambda$. □

2.4 Compact domination

Given a definable group G in a NIP theory, it is only natural to ask whether Haar measure of the compact group G/G^{00} can be lifted to an invariant measure on G. When this happens, we say that the group G/G^{00} *compactly dominates* G. Will prove later in Theorem 2.37 that this happens if and only if G admits a smooth invariant measure. In this section, we start introducing the notion of compact domination and we do some preparatory work.

Let X be a type-definable over M set and let K be a compact space. We say $f : X \to K$ is definable if for every closed set $C \subseteq K$ the preimage $f^{-1}(C)$ is type-definable. Equivalently, f is definable if the induced function $f : S_X(M) \to K$ is continuous.

Definition 2.22 (Compact domination)**.** Let X be a type-definable over M set, let h be a probability measure on compact space K, and let $f : X \to K$ be a definable function. We say that X is *compactly dominated* by (K, h, f) if for every formula $\phi(x, b) \in L(\mathcal{U})$, the f-border $\partial_f^M \phi(x; b)$ has h-measure zero (cf. Section 1.5).

We can immediately give a very simple and important example of compact domination.

Lemma 2.23. *Let X be type-definable and let μ be an M-invariant Keisler measure concentrating on X. Let μ_0 denote the associated Borel measure on $S(M)$. Then X is compactly dominated by $(S(M), \mu_0, \mathrm{tp}(\cdot/M))$ if and only if μ is smooth.*

Proof. Recall that we refer to the tp-border simply as the border of a formula. Hence, this is just Lemma 1.30. □

Lemma 2.24. *Let X be compactly dominated by (K, μ, f) and let $g : K \to H$ be a continuous map to a compact space H. Suppose that the set $F = \{x \in K : g^{-1}(g(x)) = \{x\}\}$ has μ-measure 1. Then $(H, g_*(\mu), gf)$ compactly dominates X.*

Proof. Let $\phi(x, b) \in L(\mathcal{U})$ be a formula and let $B = \partial_{gf}^M \phi(x, b)$. Let $\tilde{g} = g|_F$. Clearly $g^{-1}(B) \subseteq \tilde{g}^{-1}(B) \cup F^c$. Hence,

$$g_*(\mu)(B) = \mu(g^{-1}(B)) \leq \mu(\tilde{g}^{-1}(B) \cup F^c) = \mu(\tilde{g}^{-1}(B)),$$

where we have used that F^c has zero measure. Now, notice that $\tilde{g}^{-1}(B) = \partial_f^M \phi(x, b)$ and the latter has μ-measure zero as (K, μ, f) compactly dominates X. Thus, $g_*(\mu)(B) = 0$, as we wanted. □

We now specialize the definition of compact domination to definable groups.

Definition 2.25. If G is a definable group we say that it is compactly dominated as a group by (K, π) if K is a group, π is a group homomorphism, and G is dominated as a type-definable set by (K, h, π), where h is the Haar measure of K.

Lemma 2.26. *If G is compactly dominated as a group by some (K, π), then it is compactly dominated by $(G/G^{00}, \pi_0)$ where ϕ_0 is the canonical projection.*

Proof. Consider $\ker \pi = \pi^{-1}(e)$: it is type-definable and $G/\ker \phi = K$ so it is bounded. Then by definition G^{00} is contained in $\ker \pi$ and π factors through π_0. The compact domination condition on π_0 follows from the one on π. □

Taking in consideration the previous lemma, we will say that a group G *is compactly dominated* if it is compactly dominated by $(G/G^{00}, \pi_0)$.

Lemma 2.27. *If a group G has a smooth invariant measure $\mu \in \mathfrak{M}(\mathcal{U})$, then it is an fsg group.*

Proof. Any smooth measure is finitely satisfiable, therefore we can conclude using Proposition 2.20. In particular, any type in the support of μ is an fsg type. $\square$

Proposition 2.28. *If G is compactly dominated, then it admits a smooth invariant measure.*

Proof. Let $\pi_0 : G \to G/G^{00}$ be the standard projection map. Since G is compactly dominated, we can define a global measure μ on G by letting $\mu(X) = h(G/G^{00})$. Clearly, since h is invariant, μ is also invariant.

Now, let M be a model over which G is defined and let $f : G \to S_G(M)$ be the map $\mathrm{tp}(-/M)$. Notice that the map π_0 factors through f. Therefore, $(S_G(M), f, \mu)$ compactly dominates G. By Lemma 2.23, this means that μ is smooth. $\square$

2.5 Random elements

We now want to prove the converse of Proposition 2.28, namely that if a group has a smooth invariant measure μ then it is compactly dominated by G/G^{00}. Recall that, by Lemma 2.23, we already know that G is compactly dominated as a type-definable set by $(S_G(M), \mathrm{tp}(-/M), \mu)$. In this section, we will prove that, if the language L is countable, then the projection $\pi_0 : S_G(M) \to G/G^{00}$ is almost everywhere a bijection. With a little additional work, we will then be able to conclude by applying Lemma 2.24. The main ingredient of the proof is the notion of *random elements*, that we now introduce.

Let M be a model. We call a Borel-formula $B(x_1, \dots, x_n)$ over M, any $L_{\omega_1,\omega}$ infinitary formula in the language $L(M)$ expanded with a predicate for every open set. Consider now an elementary extension $N \succ M$. Given $a_1, \dots, a_n \in N$ it makes sense to ask if $B(a_1, \dots, a_n)$ holds. Similarly, given $p \in S^n(N)$, it makes sense to say $p \in [B(x_1, \dots, x_n)]$. Notice that any Borel-set of the Stone space $S^n(M)$ can be written as $[B(x_1, \dots, x_n)]$ for some Borel-formula B. We will often identify Borel sets with Borel formulas and, with a slight abuse of notation, we will denote them with B instead of $[B]$.

Now, let $\mu \in \mathfrak{M}(M)$ be a Borel-definable measure and let $B(x, y)$ be a Borel formula. By hypothesis, the set of types $q \in S(M)$ such that $\mu_x(B(x, q)) = 0$ is Borel-definable. Denote by $\mu^0_x.B(x, y)$ a Borel-formula defining this set. We can think of μ^0_x as a quantifier that binds the variable x. We call μ-formulas the $L_{\omega,\omega}$ formulas extended with the quantifier μ^0_x. Accordingly, we say that a set is μ-definable if it is defined by a μ-formula. Notice that

$$\text{definable sets} \subsetneq \mu\text{-definable sets} \subsetneq \text{Borel-definable sets.}$$

We now want to add a new predicate $\mathcal{B}$ to the language so that we are able to talk about borders. Given a formula $\phi(x, y) \in L(M)$, let $B_\phi(x, y) = \neg \bigvee_{\theta,\psi} \theta(x) \wedge \psi(y)$, where $\theta(x) \wedge \psi(y) \to \phi(x, y)$ or $\theta(x) \wedge \psi(y) \to \neg\phi(x, y)$. The intuition behind this definition is that $B_\phi(x, h) = \partial^M \phi(x, h)$, as one can verify. Now, write $\mathcal{B}(x, y) = \bigvee_\phi B_\phi(x, y)$ for $\phi(x, y) \in L(M)$.

We will say that a set is μ-∂-definable if it is μ-definable in the language $L(M) \cup \{\mathcal{B}\}$, where we consider $\mathcal{B}$ as a relation symbol.

Definition 2.29. A point $g \in G(\mathcal{U})$ is said to be random over Mh if it does not exist a μ-∂-definable over M set $B(x, y)$ such that $B(g, h)$ holds and $\mu(B(x, h)) = 0$.

Remark 2.30. The fact that an element g is random over Mh is completely encoded in $\mathrm{tp}(g/Mh)$. As such we could have defined the notion of a random type instead of a random element. However, to simplify the notation we prefer to talk about random elements.

In the rest of this section suppose that G is an fsg group and let μ be its unique right invariant measure. Recall that, by Proposition 2.21, the measure μ is generically stable.

Lemma 2.31. *If g is random over Mh, then $h \cdot g$ is random over Mh.*

Proof. Suppose $B(h \cdot g, h)$ holds and $\mu(B(x, h)) = 0$. Because μ is left invariant we also have $\mu(B(h \cdot x, h)) = 0$. Consider now $B'(x, y) \equiv B(y \cdot x, y)$. Then $B'(g, h)$ holds and $\mu(B'(x, h)) = 0$, however g is random over Mh. Contradiction. □

Lemma 2.32. *Let $g \in G$ be random over M and $h \in G$ be random over Mg, then g is random over Mh.*

Proof. Suppose not, then there is a μ-∂-definable set $B(x, y)$ such that $B(g, h)$ holds but $\mu(B(x, h)) = 0$. Let $C(y) = \mu^0_x.B(x, y)$. Since $C(h)$ holds and h is random, we must have $\mu(C(y)) > 0$. Without loss of generality, restrict $B(x, y)$ to $B(x, y) \cap C(y)$. Clearly, we have

$$\mu_x \otimes \mu_y(B(x, y)) = \int_{q \in S(M)} \mu_x(B(x, q_y))\, \mathrm{d}\mu_y = 0.$$

Since μ is generically stable, it commutes with itself. Therefore, we also have

$$\mu_y \otimes \mu_x(B(x, y)) = \int_{p \in S(M)} \mu_y(B(p_x, y))\, \mathrm{d}\mu_x = 0.$$

Since the integral of a positive function is zero if and only if the function is zero almost everywhere, we can conclude that the set $\mu^0_y.B(x, y)$ has measure 1, so g must lie in it. Hence, we have $\mu_y(B(g, y)) = 0$, contradicting the fact that h is random over Mg. □

Lemma 2.33. *If $g \in G(\mathcal{U})$ is random over Mh and $h \in G^{00}$, then* $\mathrm{tp}(g/Mh) = \mathrm{tp}(h \cdot g/Mh)$.

Proof. Let $\phi(x, y) \in L(M)$ be a formula. We claim that $\mu_x(\phi(x, h) \leftrightarrow \phi(h \cdot x, h)) = 1$. In fact, suppose that its complementary had positive μ-measure. Then there would be a global fsg type p such that $p \vdash \neg(\phi(x, h) \leftrightarrow \phi(h \cdot x, h))$. Since, by Lemma 2.14, fsg types are G^{00}-invariant, this is a contradiction.

As g is random over Mh and the set $g \models \phi(x, h) \leftrightarrow \phi(h \cdot x, h)$ has μ-measure 1, we must have $g \models \phi(x, h) \leftrightarrow \phi(h \cdot x, h)$. Since this is true for all formulas $\phi(x, h)$, we can conclude $\mathrm{tp}(g/Mh) = \mathrm{tp}(h \cdot g/Mh)$. □

Lemma 2.34. *Suppose that L is countable and that μ is a smooth measure. Let $g \in G(\mathcal{U})$ be random over Mh and let $g' \in G(\mathcal{U})$ be an element such that* $\mathrm{tp}(g/M) = \mathrm{tp}(g'/M)$*, then* $\mathrm{tp}(g/Mh) = \mathrm{tp}(g'/Mh)$.

Proof. Let $\mathcal{B}(x, y)$ be as in the definition of μ-∂-definable set. Recall that $\mathcal{B}(x, h) = \bigcup_\phi \partial\phi(x, h)$. Then, since μ is smooth and the language is countable, $\mathcal{B}(x, y)$ is the union of countably many sets of zero measure, therefore it has zero measure. As g is random over Mh, g does not lie in that set. The thesis follows by definition of border of a formula. □

Lemma 2.35. *Suppose that L is countable and that μ is a smooth measure. Then the subset of the random types of $S(M)$ has μ-measure 1.*

Proof. Notice that, since the language is countable, there are only countably many μ-∂-definable sets. Therefore the union of all μ-∂-definable sets of μ-measure zero is still Borel and it has μ-measure zero. To conclude, notice that any type not contained in this union is random. □

Lemma 2.36. *Suppose that L is countable and that μ is a smooth measure. Let $\pi_0 : S_G(M) \to G/G^{00}$ be the projection map. Then, the subset $\{p \in S_G(M) : \pi_0^{-1}(\pi_0(p)) = \{p\}\}$ has μ-measure 1. That is, π_0 is almost everywhere a bijection.*

Proof. Let $g, g' \in G(\mathcal{U})$ be in the same coset of G^{00} and suppose that g is random over M. Let $h \in G(\mathcal{U})$ be random over Mgg'. By Lemma 2.33, the type $\mathrm{tp}(h/Mgg')$ is $g^{-1} \cdot g'$ invariant, hence $\mathrm{tp}(g'^{-1} \cdot h/Mgg') = \mathrm{tp}(g^{-1} \cdot h/Mgg')$ and in particular, by taking the inverse,

$$\mathrm{tp}(h^{-1} \cdot g/M) = \mathrm{tp}(h^{-1} \cdot g'/M).$$

By Lemma 2.32, g is random over Mh so, by Lemma 2.31, $h^{-1} \cdot g$ too is random over Mh. We can then apply Lemma 2.34 to get

$$\mathrm{tp}(h^{-1} \cdot g/Mh) = \mathrm{tp}(h^{-1} \cdot g'/Mh).$$

This implies $\mathrm{tp}(g/Mh) = \mathrm{tp}(g'/Mh)$, and more so $\mathrm{tp}(g/M) = \mathrm{tp}(g'/M)$. □

Theorem 2.37. *If a definable group G admits a smooth left invariant measure, then it is compactly dominated.*

Proof. First, we show that we can reduce to the case where L is countable. Let $\bar{L} \subseteq L$ be a countable language over which G is defined and such that the restriction of μ to $\bar{L}$ is still smooth. Let $G^{00}_{\bar{L}}$ be the connected component of G relative to this language. Clearly, $G^{00}_{\bar{L}} \subseteq G^{00}$. Now, suppose that G is compactly dominated by $G/G^{00}_{\bar{L}}$. Then more so, by Lemma 2.26, it is compactly dominated by G/G^{00}.

We can therefore assume L to be countable. Let $\pi_0 : S(M) \to G/G^{00}$ and $\pi : S(M) \to G/G^{00}$ be the standard projection maps, and let $f : G \to S_G(M)$ be the map $\mathrm{tp}(-/M)$. Notice that $\pi = \pi_0 f$. Notice also that $\pi_{0*}(\mu_0)$ is a left invariant probability measure, hence it must coincide with the unique Haar measure h of G/G^{00}. By Lemma 2.23, since μ is smooth, G is compactly dominated by $S(M)$. By Lemma 2.36, using the fact that L is countable, the map π_0 is almost everywhere a bijection between $S(M)$ and G/G^{00}. Therefore, by Lemma 2.24, G is compactly dominated by $(G/G^{00}, \pi, \pi_{0*}(\mu_0))$. Lastly, by the previous remark, $\pi_{0*}(\mu_0) = h$. Therefore, G is compactly dominated as a group by G/G^{00}. □

Finally, we explicitly state and prove the converse of the previous theorem.

Theorem 2.38. *Let G be a definable group in an NIP theory. Then G is compactly dominated if and only if G admits a smooth global invariant measure.*

Proof. If G is compactly dominated, then, by Proposition 2.28, it admits a smooth global invariant measure. Conversely, if G admits a smooth global invariant measure then by the previous theorem it is compactly dominated. $\square$

Chapter 3

O-minimal theories

3.1 O-minimal structures

Definition 3.1. We say that a structure $M = (M, <, \ldots)$ is o-minimal if every definable subset of M is a finite union of points and intervals, where by interval we mean a set of the form (a, b) with $a, b \in M \cup \{\pm\infty\}$.

We say that a theory T is o-minimal if every model $M \models T$ is o-minimal.

Example 3.2. By quantifier elimination, the theory of dense linear orders and the theory of discrete linear orders are o-minimal. In particular, $(\mathbb{Q}, <)$ and $(\mathbb{Z}, <)$ are o-minimal. However, adding structure to those models does not preserve o-minimality. For example, $(\mathbb{Z}, <, +)$ is not o-minimal, as the set of even numbers it is not finite union of points and intervals but it is nonetheless definable, for example using the formula $\phi(x) \equiv \exists y(x = y + y)$. Notice that, by quantifier elimination, $(\mathbb{Z}, <, +)$ is still an NIP structure [Mar02, Corollary 3.1.21].

Likewise, the field of rational numbers $(\mathbb{Q}, <, +, \cdot)$ is not o-minimal. For an easy proof, consider the set $\{x \in \mathbb{Q} : x^2 > 2 \wedge x > 0\} = (\sqrt{2}, +\infty)$. This is a definable set, however $\sqrt{2} \notin \mathbb{Q}$ (recall that we ask for the extrema of the intervals to be in the model). Moreover, Julia Robinson proved that $\mathbb{Z}$ is definable in $(\mathbb{Q}, <, +, \cdot)$ [Rob49]. Therefore, $\mathbb{Q}$ fails critically to be o-minimal as it can interpret Peano's arithmetic.

We now introduce our most important example of an o-minimal theory.

Theorem 3.3. *Let $\mathcal{R}$ be a real closed field. Then $\mathcal{R}$ is o-minimal. In particular, the structure $(\mathbb{R}, <, +, \cdot)$ is o-minimal.*

Proof. Let $\phi(x, b)$ be a definable set. By Tarski's quantifier elimination theorem for RCF, we have

$$\phi(x, y) \equiv \bigwedge_{i<n} \bigvee_{j<m_n} p_{i,j}(x, y) \leq 0,$$

where each $p_{i,j}(x, y)$ is a polynomial. It now suffices to notice that a boolean combination of sets of the form $p(x, b) \leq 0$ is always a finite combination of points and interval. $\square$

It is a very important theorem by Wilkie that the field of real numbers expanded with the exponential function $(\mathbb{R}, <, +, \cdot, \exp)$ is o-minimal [Wil96]. This result still holds if we also add all the real analytical functions restricted to a compact box $[a, b]^n$ [vdDM94]. Notice that this last theorem fails if we do not restrict ourselves to a closed and bounded box. Consider for example the structure $(\mathbb{R}, <, +, \cdot, \sin)$: the set $Z = \{x \in \mathbb{R} : \sin(x) = 0\}$ is definable and it is not a finite union of points and intervals.

By Theorem 3.3 we know that real closed fields are o-minimal. It is interesting to ask what other algebraic structures can be o-minimal. We now present some strong results in this direction.

Theorem 3.4 ([PS86, Theorem 2.1])**.** *Let $M = (M, \leq, \dots)$ be a totally ordered group (i.e. such that $x \leq y \Rightarrow uxv \leq uyv$). Then M is o-minimal if and only if M is abelian and divisible.*

Proof. First we prove that if M is o-minimal then it is abelian and divisible. This will follow easily by proving that the only definable subgroups of M are $\{e\}$ and M itself. Assume by contradiction that $G < M$ is a nontrivial subgroup. First we prove that G must have a supremum. Assume not: then by o-minimality it should contain the interval $[g, +\infty)$ for some $g \in G$. Then, we would have $G = M$, as for each $e < m \in M$ we have $gm \in G$ and therefore $g^{-1}(gm) = m \in G$.

Therefore, we may assume that G has a supremum $s > e$. We will prove that $s \in G$. In conjunction with the fact that $s^2 \geq s$, this imply $s = e$ and therefore $G = \{e\}$. To prove $s \in G$, notice that, by definition of supremum, we can find an element $e < g \in G$. It follows that $g^{-1}s < s$. Again by definition of supremum, there is some $h \in G$ such that $g^{-1}s < h < s$. But then $s < gh$ and therefore $s = gh \in G$.

Now that we know that the only definable subgroups are trivial, the rest is easy. Fix any $e \neq g \in G$. Its centralizer $C(g)$ is a definable subgroup of M and $C(g) \neq \{e\}$ as $g \in C(a)$. Therefore, $C(g) = M$. As this holds for any $g \in M$, we can conclude that M is abelian.

Since M is abelian, the $M^n = \{g^n \in M : g \in M\}$ is a definable subgroup of M. As $g^n > g$ for any $e \neq g \in M$, we also know that $M^n \neq \{e\}$. Therefore, for any $n > 0$, we have $M^n = M$. Hence, M is divisible.

The converse is an easy consequence of the fact that the theory of divisible ordered abelian groups admits quantifier elimination in the language $\{+, -, \leq, 0\}$ [Mar02, Corollary 3.1.17]. □

Lemma 3.5. *An ordered field M is o-minimal if and only if it is real closed.*

Proof. We have already seen a direction in Theorem 3.3. For converse, it suffices to prove the intermediate value theorem for definable continuous map, so that we will know that any polynomial that changes sign has a zero. We will obtain it later as a consequence of the fact that the image of a definably connected set is definably connected (cf. Section 3.5). □

Theorem 3.6 ([PS86, Theorem 2.3])**.** *An ordered ring M is o-minimal if and only if it is a real closed field.*

Proof. By the previous lemma, it suffices to show that ordered rings are in fact fields. Let $M^+ = \{m \in M : m > 0\}$, we will show that M^+ is a group under multiplication. The only nontrivial part is showing that an element $r \in M^+$ has an inverse. To do this, notice that $rM = \{r \cdot m : m \in M\}$ is a additive subgroup of M. By the proof of Theorem 3.4, we know that $rM = M$. Hence, there is an $m \in M$ such that $r \cdot m = 1$.

Now restrict to the structure M^+. It is easy to argue that M^+ is still o-minimal since it is a definable convex subset of M. By Theorem 3.4, M^+ must be an abelian group. This concludes the proof. □

We close this section with the fundamental remark that o-minimal theories are NIP.

Proposition 3.7. *Let T be an o-minimal structure. Then T is NIP.*

Proof. Proceed exactly as in Example 1.7, using o-minimality in place of quantifier elimination. □

3.2 Structure theorems for definable sets and functions

In this section, we will present some of the most important decomposition theorems for definable sets and functions in o-minimal structures.

First, we would like to be able to talk about continuous functions. Notice that on an o-minimal structure M we can always consider the interval topology, that is the topology that has the open intervals as a basis of open sets. On the product set M^n we shall consider the product topology. Once we have fixed a topology, it makes sense to say that a definable function is continuous at a point x. The following simple lemma will often be used implicitly.

Lemma 3.8. *Let $f : M \to M$ be a definable function and let $X = \{x \in M : f \text{ is continuous at } x\}$. Then X is definable.*

Many theorems about o-minimal structures state that a definable set can be divided in finitely many parts in such a way that each part is well-behaved in some sense. The Monotonicity Theorem is one of the most basic and important theorems in this direction.

Theorem 3.9 (Monotonicity Theorem)**.** *Let $M = (M, \leq, \dots)$ be o-minimal and let $f : M \to M$ be definable. There are $a_0 = -\infty < a_1 < \dots < a_n = +\infty$ such that for each i, f is constant or continuous and strictly monotone on (a_{i-1}, a_i).*

The proof is based on the following lemma.

Lemma 3.10. *Let $f : I \to M$ be a definable function on an interval I. Then*

1. *There is a subinterval of I where f is constant or injective.*
2. *If f is injective, then f is strictly monotone on a subinterval of I.*
3. *If f is strictly monotone, then f is continuous on a subinterval of I.*

Sketch of the proof. We only prove the first point. The proof of the other points is similar, although lengthy, and can be found in [vdD98].

Suppose that $f^{-1}(y)$ was infinite for some $y \in M$. Then, by o-minimality, it would contain an interval $J \subseteq f^{-1}(y)$ and f would be constant on J. We can therefore assume that each $y \in M$ has finite preimage. Define $g : M \to I$ by

$$g(y) = \min\{x \in I : f(x) = y\}.$$

Notice that g is injective, hence $g(M) \subseteq I$ is infinite. Let J be an interval in $g(M)$. Since g is an inverse of f on J, f is injective there. □

Proof of Theorem 3.9. Define the set

$$X = \{x \in M : f \text{ is stricly monotone and continuous near } x\},$$

where by "near x" we mean in an open interval containing x. Let $Y = M \setminus X$. Suppose for the moment that Y is finite and define:

$$\begin{aligned} X_= &= \{x \in X : f \text{ is constant near } x\} \\ X_+ &= \{x \in X : f \text{ is strictly increasing and continuous near } x\} \\ X_- &= \{x \in X : f \text{ is strictly decreasing and continuous near } x\}. \end{aligned}$$

By o-minimality, we can find some points $a_0 = -\infty < a_1 < \ldots < a_n = +\infty$ such that each interval (a_i, a_{i+1}) is entirely contained in $X_=$, X_+, or X_-. Notice that the points of Y will be among the a_i.

Clearly, if $(a_i, a_{i+1}) \subseteq X_=$ then f is constant on (a_i, a_{i+1}). Now, suppose that $(a_i, a_i + 1) \subseteq X_+$ and let $x \in (a_i, a_{i+1})$. We want to show that for any $y > x$ we have $f(y) > f(x)$. But this is clear, since by o-minimality the set of $y > x$ such that the condition holds will have a supremum by o-minimality and by definition of X_+ that supremum must be a_{i+1}. The same reasoning works for X_-. Therefore, we have proved the theorem assuming the fact that Y is finite.

Hence, all it remains to show is that Y must be finite. Suppose it were not, then by o-minimality it would contain an interval I. By applying in succession the three points of the previous lemma, we can find a subinterval $J \subseteq I$ such that f is either constant or strictly monotone and continuous on J. However, this implies $J \subseteq X$. Contradiction. □

Corollary 3.11. *Assume that $f : [a, b) \to M$ is a definable function, then $\lim_{x \to b^-} f(x)$ exists in $M \cup \{\pm\infty\}$.*

Proof. By the Monotonicity Theorem, there is some $a < b' < b$ such that f is monotone on (b', b). We can then conclude, as every monotone function admits limits. □

If M expands a field, then it makes sense to define the derivative of a definable function $f : M \to M$ at a point x as

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}.$$

Notice that by Corollary 3.11 the limit always exists, although it can be $\pm\infty$. However, it is not difficult to prove that a definable function in one variable is differentiable in all but finitely many points [vdD98, Chapter 7, (2.5)]. Therefore, we can improve the monotonicity theorem and also ask for f to be derivable on each interval.

We now want to generalize these structure theorems to sets and functions in dimension greater than one, leading us to the single most important theorem in the study of o-minimal structures: the *cell decomposition theorem.* Cells are topologically trivial sets that generalize the role of points and intervals in dimension greater than one:

Definition 3.12. Let $(\varepsilon_1, \ldots, \varepsilon_m)$ be a sequence of zeros and ones of length m. We define a $(\varepsilon_1, \ldots, \varepsilon_m)$*-cell* by induction as follow:

(i) A (0)-cell is a point in M. A (1)-cell is an interval $(a, b) \subseteq M$.

(ii) Suppose we have defined the $(\varepsilon_1, \ldots, \varepsilon_m)$-cells.

- A $(\varepsilon_1, \ldots, \varepsilon_m, 0)$-cell is the graph $\Gamma_f \subseteq C \times M$ where C is a $(\varepsilon_1, \ldots, \varepsilon_m)$-cell and $f : C \to M$ is a continuous function.
- A $(\varepsilon_1, \ldots, \varepsilon_m, 1)$-cell is the region $(f, g)_C = \{(x, y) \in C \times M : f(x) < y < g(x)\}$ where C is a $(\varepsilon_1, \ldots, \varepsilon_m)$-cell and $f, g : C \to M$ are continuous function such that $f < g$ (we allow $f = -\infty$ or $g = +\infty$).

If C is a $(\varepsilon_1, \ldots, \varepsilon_m)$-cell, we say that the *dimension of* C *as a cell* is $\varepsilon_1 + \ldots + \varepsilon_n$.

Suppose that C is a $(\varepsilon_1, \ldots, \varepsilon_m)$-cell and let $i_1, \ldots, i_d$ be the coordinates such that $\varepsilon_{i_k} = 1$. Notice that by definition d coincides with the dimension of the cell C. Now, let $\pi : M^m \to M^d$ be the projection onto those coordinates, it is easy to prove by induction that $\pi(C) \subseteq M^d$ is an open cell and that $\pi|_C$ is an homeomorphism of C onto $\pi(C)$. Therefore, cells are topologically trivial. However, notice that their embedding in M^n can be quite complicated. For example, in [BF09, 6.1], they give an example of a 2-dimensional semialgebraic cell C in $\mathbb{R}^4$, whose closure is homotopic to a circle.

Lastly, we introduce the notion of a *decomposition of* M^n as follow:

(i) a decomposition of M is any partition of the form

$$\{(-\infty, a_1), (a_2, a_3), \ldots, (a_k, +\infty), \{a_1\}, \ldots, \{a_k\}\},$$

where $a_1 < \ldots < a_k$ are points in M.

(ii) a decomposition of M^{n+1} is a partition $M^{n+1} = C_1 \cup \ldots \cup C_k$ into finitely many cells such that the set of projections $\{\pi(C_i) : i \leq k\}$ is a decomposition of M^n.

We are now ready to state the theorem.

Theorem 3.13 (Cell Decomposition Theorem)**.** *Let M be an o-minimal expansion of a densely ordered set, let $X \subseteq M^n$ and let $f : X \to M^m$ be a definable function. There is a decomposition of M^n into finitely many cells $C_1, \ldots, C_k$ which is compatible with X (i.e. each C_i is either disjoint from X or contained in X), such that $f|_{C_i}$ is continuous for each i. If M extends a field, we can moreover ask for $f|_{C_i}$ to be C^k for any $k < \omega$.*

The proof of the theorem is a lengthy induction on the dimension of the definable set X, having the monotonicity theorem as the basic case [vdD98, Chapter 3, (2.11)]. We should also mention that the version we have stated is very basic, and a number of improvements can be made. For example we can impose additional monotonicity conditions [vdD98, Chapter 3, (2.19)]. This version, however, will be sufficient for this work.

We conclude this section with an important corollary of the cell decomposition theorem, the *uniform boundedness theorem* for definable family of sets, from which we derive some theoretical and practical consequences.

Theorem 3.14 (Uniform boundedness)**.** *If M is an o-minimal L-structure and $\phi(x, y)$ is an L-formula, then there is some $K \in N$ such that for all $|x|$-tuples a from M, the subset of $M^{|y|}$ defined by $\phi(a, y)$ is a union of at most K cells.*

Proof. Let $Z = \phi(M^{|x|}, M)$ be the set defined by ϕ and let $Z = C_1 \cup \ldots \cup C_K$ be a cell decomposition of Z. Define $Z_a = \{y \in M : (a, y) \in Z\}$. We want to show that Z_a is the union of at most K cells. But in fact $Z_a = (C_1)_a \cup \ldots \cup (C_K)_a$ and it is clear from the definition of a cell that each $(C_i)_a$ is still a cell. □

We now prove that if a structure M is o-minimal then every elementary equivalent structure $N \equiv M$ is also o-minimal (and therefore $\mathrm{Th}(M)$ is o-minimal). In analogy with the notion of strong minimality, we refer to this property by saying that o-minimal structures are strongly o-minimal.

Corollary 3.15. *If M is an o-minimal structure and N is elementary equivalent to M, then also N is o-minimal.*

Proof. Let $\phi(x, y) \in L$, with $|x| = 1$, be a formula. By the previous theorem there is some $K \in \mathbb{N}$ such that for any $b \in M$ the formula $\phi(x, b)$ defines a union of at most K many points and intervals. As this fact is expressible by a first order statement, the same must hold for N. In particular, this means that for each $b \in N$, the formula $\phi(x, b)$ defines a finite union of points and intervals in N, as we wanted. □

Lastly, we present an application of o-minimality to give uniform bounds in number theory.

Example 3.16 (Generalize Descartes' sign rule)**.** Recall that given a polynoimial p in the form

$$p(x) = \sum_{i=1}^{K} a_i x^{n_i},$$

Descartes' sign rule tells us that the number of real zeros of $p(x)$ is less than $2K$. In particular, the bound does not depend on the coefficient a_i nor from the exponents n_i. We now prove a generalisation of this statement for polynomials in two variables. Let

$$q(x, y) = \sum_{i=1}^{K} \sum_{j=1}^{K} a_{i,j} x^{n_i} y^{n_j},$$

and let $Z(q) \subseteq \mathbb{R} \times \mathbb{R}$ be the zero set of $q(x, y)$. We claim that the number of connected components of $Z(q)$ is bounded by a number $N(K)$ depending only on K. To see this, substitute $x = e^t$ and $y = e^s$. After the substitution we have

$$q(s, t) = \sum_{i=1}^{K} \sum_{j=1}^{K} a_{i,j} \exp(sn_i + tn_j).$$

Now $q(s,t)$ is a definable function in the o-minimal theory $(\mathbb{R}, <, +, \cdot, \exp)$ having $a_{i,j}$,n_i as parameters. Notice that cells are connected, as every cell is homeomorphic to $\mathbb{R}^d$ for some d. Hence, by Theorem 3.14, for any choice of these parameters the zero set of $q(s,t)$ has at most $N(K)$ many connected components, where $N(K)$ depends only on K. Knowing this about the zero set of $q(s,t)$ easily let us conclude the same for the zero set of $q(x,y)$.

3.3 Geometries

One of the main advantages of working with o-minimal theories instead of NIP theories is the one is the existence of a very well behaved concept of dimension given by algebraic independency. In this section we define and prove some of the main properties of this concept dimension, showing that o-minimal theories are pregeometries. This result will then be further strenghtened, by showing that o-minimal theories are in fact geometries.

Recall that we call a theory T *pregeometric* if every model M of T has the *exchange property*, that is if for all $a, b \in M$ and $A \subseteq M$ we have

$$b \in \operatorname{acl}(Aa) \iff a \in \operatorname{acl}(Ab).$$

Suppose now that T is a pregeometric theory, and let $\bar{a} \in \mathcal{U}$ be a tuple. We define the *dimension of the tuple* $\bar{a}$, denoted $\dim(\bar{a}/A)$, as the maximum cardinality of an algebraic independent subtuple of $\bar{a}$. Reasoning as in linear algebra and using the exchange property, it is easy to see that every maximal independent subtuple has the same cardinality.

Now, let X be an A-definabile set. We define the *dimension of* X as

$$\dim(X) = \max\{\dim(\bar{a}/A) : \bar{a} \in X(\mathcal{U})\},$$

where A is any small set of parameters over which X is defined (it is not difficult to show that the actual choice of A is ininfluent). Notice that if $X \subset \mathcal{U}^n$, then $\dim(X) \leq n$ so the dimension of a set is always finite. We say that a point $\bar{a} \in X$ is generic for X if $\dim(X) = \dim(\bar{a}/A)$.

Example 3.17. Any strongly minimal theory is a pregeometric theory [Mar02, Lemma 6.1.4]. In particular, algebrically closed fileds are pregeometries. In this case the dimension of an element over a set of parameters A coincides with the degree of trascendence $\operatorname{trdeg}_A(\bar{a})$ of $\bar{a}$ over A.

This notion of dimension is very well behaved, and several intuitive properties hold.

Proposition 3.18. *Let T be a pregeometric theory and let $A \subseteq \mathcal{U}$ be a small set of parameters. Let $\bar{a} \in \mathcal{U}$ a be tuple. Then:*

1. *If $A \subseteq B$, then* $\dim(\bar{a}/A) \geq \dim(\bar{a}/B)$
2. $\dim(\bar{a}\bar{b}/A) = \dim(\bar{a}/A\bar{b}) + \dim(\bar{b}/A)$

Let X, Y be A-definable sets. Then:

3. *Let $f : X \to Y$ be an A-definable surjective function and suppose* $\dim(f^{-1}(x)/Ax) = k$ *for all $x \in X$. Then* $\dim(X) = \dim(Y) + k$.

4. $\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}$

Proof. Points 1, 2 and 4 follow almost immediately from the definitions.

We prove point 3. Let $\bar{b}$ be a generic element of Y, and choose $\bar{a} \in f^{-1}(\bar{b})$ generic over $A\bar{b}$. By point 2, we have $\dim(\bar{a}\bar{b}/A) = \dim(\bar{a}/A\bar{b}) + \dim(\bar{b}/A)$. Now, notice that $\dim(\bar{a}\bar{b}/A) = \dim(\bar{a}/A)$, as $\bar{b}$ is A-definable from $\bar{a}$. Therefore we have

$$\dim(X) \geq \dim(\bar{a}/A) = k + \dim(Y).$$

For the other inequality, let $\bar{a}$ be generic for X and let $\bar{b} = f(\bar{a})$. Then

$$\dim(X) = \dim(\bar{a}\bar{b}/A) = \dim(\bar{a}/A\bar{b}) + \dim(\bar{b}/A) \leq k + \dim(Y),$$

where we used $\dim(\bar{b}/A) \leq \dim(Y)$ and $\dim(\bar{a}/A\bar{b}) \leq k$. □

Remark 3.19. As a consequence of point 3 above, we obtain that $\dim(X \times Y) = \dim(X)\dim(Y)$. Moreover, if there is a definable bijection between two sets X and Y, then $\dim(X) = \dim(Y)$.

We now show that an o-minimal theory is always a pregeometry, that is every o-minimal structure has the exchange property.

Theorem 3.20. *Let M be an o-minimal structure. Then M has the exchange property.*

Proof. Let M be a model of an o-minimal theory. Notice that, since M has an order, $\mathrm{acl}(A) = \mathrm{dcl}(A)$. Therefore we need to prove that, if $b \in \mathrm{dcl}(Aa)$ and $b \notin \mathrm{dcl}(A)$, then $a \in \mathrm{dcl}(Ab)$. The fact that $b \in \mathrm{dcl}(Aa)$ means precisely that there is an A-definable function $f : M \to M$ such that $f(a) = b$. By the Monotonicity Theorem, since $b \notin \mathrm{acl}(A)$, f must be strictly monotone or constant in an interval containing a. If f were costant near a, then b would be A-definabile, as f is. Therefore f is strictly monotone. Let g be the inverse of f. Then $a = g(b)$ and thus $a \in \mathrm{dcl}(Ab)$. □

Remark 3.21. Let C be a cell. Recall that C is homeomorphic to M^d, where d is the dimension of C as a cell. Then, by Remark 3.19, we have $\dim(C) = \dim(M^d) = d$, that is the dimension of C as a cell coincide with its definable dimension. Hence, using the cell decomposition theorem, we can conclude that the dimension of a set coincides with the maximal dimension of one of its cell.

The cell decomposition theorem allows us to strengthen further these results and to prove that o-minimal structures are in fact geometries.

Definition 3.22. A pregeometric theory T is geometric if one of the followings holds:

(i) For every definable function $f : X \to Y$ and $k \in \mathbb{N}$, the set $\{y \in Y : \dim(f^{-1}(y)) = k\}$ is definable.

(ii) For every L-formula $\phi(x, \bar{y})$ there is $n \in \mathbb{N}$ such that, in every model, $\exists^\infty x \phi(x, \bar{y}) \iff \exists^{\geq n} x \phi(x, \bar{y})$, where $\exists^\infty$ means "there are infiniteley many".

We usually refer to property (i) as *definability of dimension* and to property (ii) as *uniform boundedness.*

Proposition 3.23. *O-minimal structures are geometric.*

Proof. To prove definability of dimension, let $f : X \to Y$ be a definable function and consider a cell decomposition of its graph $\Gamma(f)$. The uniform boundedness property follows almost directly from Theorem 3.14. □

3.4 Elimination of imaginaries

In this section we prove that an o-minimal structure that extends a (necessarily divisible) ordered group has definable Skolem functions and elimination of imaginaries.

Lemma 3.24. *Let $M = (M, \leq, +, ...)$ be an o-minimal expansion of an ordered group. Then we can definably pick an element $e(X) \in X$ from each nonempty definable set $X \subseteq M^m$.*

Proof. We proceed by induction. Let $1 \in M$ be a positive element and let $X \subseteq M$ be definable and nonempty. If X has a least element, let $e(X)$ be this element. If X does not have a least element, let (a, b) be its left-most interval, that is we let $a = \inf(X)$ and $b = \sup\{x \in M : (a, c) \subseteq M\}$. We put

$$e(X) = \begin{cases} 0 & \text{if } a = -\infty, b = +\infty, \\ b - 1 & \text{if } a = -\infty, b \in M, \\ a + 1 & \text{if } a \in M, b = +\infty, \\ (a + b)/2 & \text{if } a, b \in R. \end{cases}$$

Now, we assume the lemma for m and we prove it for $m+1$. Let $X \subseteq M^{m+1}$ and let $\pi : M^{m+1} \to M^m$ be the projection on the first m coordinates. Since $\pi(X) \subseteq M^m$, we may assume $a = e(\pi(X))$ to be already defined. Then $X_a \subseteq M$ and we put $e(X) := (a, e(X_a))$. □

Theorem 3.25. *Let $M = (M, \leq, +, ...)$ be an o-minimal expansion of an ordered group and let $1 \in M$ be a positive element. Let $Z \subseteq M_k \times M_n$ be $\emptyset$-definable and let $X \subseteq M_k$ be its projection onto the first k coordinates. For $x \in M_k$ we write $Z_x = \{y \in M_n \mid (x, y) \in Z\} \subseteq M_n$. Then there is a map $f : X \to M_n$ definable with parameters from $\{1\}$ such that*

1. *Z contains the graph of f, that is $f(x) \in Z_x$ for all $x \in X$, and*

2. *for all $x_1, x_2 \in X$ with $Z_{x_1} = Z_{x_2}$ we have $f(x_1) = f(x_2)$.*

Proof. For $a \in X$, we define $f(a) = e(Z_a)$. It is easy to see that this definition satisfy all the requirements. □

Recall that we say that complete theory T has elimination of imaginaries if, in any model of T, for every definable equivalence relation E on a definable set X there is a definable set Y and a definable surjective function $f : X \to Y$ (over the same parameters) such that xEy if and only if $f(x) = f(y)$. If this is the case, we can identify X/E with Y and consider it as a definable object.

Theorem 3.26. *Assume $M = (M, <, +, \dots)$ is an o-minimal expansion of a divisible ordered group. Then* $\mathrm{Th}(M)$ *has elimination of imaginaries.*

Proof. Let E be a definable equivalence relationship on the definable set X. We can think of E as the subset of $X \times X$ given by $E = \{(x, y) \in X \times X : xEy\}$. Applying the Theorem 3.25 to E, we obtain a function $f : X \to X$ such that $f(x_1) = f(x_2)$ if and only if $x_1 E x_2$. Finally, let $Y = f(X)$ in the definition of elimination of imaginaries. □

Elimination of imaginaries is extremely important in the study of definable groups since it tells us that the quotient G/H of a definable group G by a definable subgroup H is a definable object. However, it should be noted that in general elimination of imaginaries can fail in o-minimal structures that do not expand an ordered group [Joh14]. Fortunately, a definable group G always admits elimination of imaginaries with the structure induced by M [Edm03, Theorem 7.2], hence we can always consider G/H to be definable.

Remark 3.27. By additivity of dimension and elimination of imaginaries, we have $\dim(G) = \dim(G/H) + \dim(H)$.

3.5 Point-set topology in o-minimal structures

Recall that on an o-minimal structure M we consider the topology that has the open interval as a basis. When $M \neq \mathbb{R}$, this topology is rather ill-behaved when studied using classical notions. For example one can prove that if M extends a real closed field, then M is totally disconnected unless $M = \mathbb{R}$.

However, these problems disappear if we restrict to definable sets and definable maps between them. For example, we will say that a definable set is *definably connected* if it cannot be written as a disjoint union of definable open sets. It is straightforward to show that the connected subsets of any o-minimal structure M are precisely the intervals, and that the image of a definably connected set under a definable continuous map is definably connected.

It is not equally easy to give a notion of compactness. The naïve approach would be to say that a set is definably compact if we can extract a finite subcover from every cover by definable open subsets. However, this approach does not work well. For example, let $\mathcal{R}$ be a nonstandard expansion of $\mathbb{R}$ and let $\varepsilon \in \mathcal{R}$ be an infinitesimal element. Then the open cover $\{(a - \varepsilon, a + \varepsilon)\}_{a \in [0,1]}$ of the closed interval $[0, 1] \subset \mathcal{R}$ does not admit a finite subcover. Instead, we give the following definition.

Definition 3.28. We say that a definable set X is definably compact if for every $\delta > 0$ and every definable curve $\gamma : (0, \delta] \to X$, the limit $\lim_{t\to 0} \gamma(t)$ is in X.

If the structure has definable choice, e.g. if we are in an o-minimal expansion of an ordered group, we can characterize the definably compact sets as the closed and bounded definable sets. We use the following very useful lemma.

Lemma 3.29 (Curve selection lemma)**.** *Let M be an o-minimal structure that has definable choice. If $X \subseteq M^n$ is definable and $y \in \bar{X} \setminus X$, then there is a continuous definable map $\gamma : (0, \delta] \to X$ for some $\delta > 0$ such that* $\lim_{t\to 0} \gamma(t) = y$.

Proof. Let $Z = \{(x, \varepsilon) \in X \times M : \varepsilon > 0 \text{ and } \|x - y\| < \varepsilon\}$. Using definable choice on Z, we can find a function $f : M \to X$ such that $\|f(\varepsilon) - y\| < \varepsilon$. By the monotonicity theorem we can find an interval $(0, \delta]$ on which every component of f is continuous. Then, by definition of Z, the limit $\lim_{t\to 0} \gamma(t)$ exists and it must coincide with y. □

Proposition 3.30. *Suppose that M is an o-minimal expansion of an ordered field, then a subset S of M^n is closed and bounded in M^n if and only if it is definably compact.*

Proof. By the previous lemma, a definably compact set is closed. Now, suppose by contradiction that it is unbounded. Consider the set $Z = \{(x, r) \in X \times M : \|x\| > r\}$. By elimination of imaginaries applied to the set Z, we can find a function $f : [0, +\infty) \to S$ such that $\lim_{x\to+\infty} f(x)$ is not in S. Using the monotonicity lemma and rescaling the domain we reach a contradiction.

Conversely, suppose that S is closed and bounded. Then existence of one sided limits easily imply that the set is definably compact. □

Now, we go back to definably connectedness, and we deduce an interesting consequence of the cell decomposition theorem. In particular, this will be useful in the study of definable groups.

Proposition 3.31. *Let M be an o-minimal structure and $X \subseteq M$ a definable set. Then X has finitely many definably connected components.*

Proof. Let $X = C_1 \cup \ldots \cup C_n$, where each C_i is a cell. Recall that C_i is definably homeomorphic to M^d, for $d = \dim(C_i)$, so in particular it is definably connected. Hence, each C_i is entirely contained in exactly one definably connected component of X and therefore there can be at most n connected components. □

3.6 Euler characteristic

As we have seen, the dimension of a definable set is invariant for definable bijection. In this section, we introduce another, more subtle, invariant: the *Euler characteristic* of a definable set.

Let C be a cell, we call *Euler characteristic* of C the number $E(C) = (-1)^{\dim(C)}$. Now, let $X \subseteq M^n$ be a definable set and let $\mathcal{D}$ be a cell decomposition of X. We call *Euler characteristic of X relative to the cell decomposition $\mathcal{D}$* the number

$$E_{\mathcal{D}}(X) = \sum_{C \in D} E(C) = \sum_{d<n} (-1)^d k_d,$$

where k_i is the number of cell in $\mathcal{D}$ of dimension d. Equivalently, $E_{\mathcal{D}}(X)$ is the number of cells in $\mathcal{D}$ of even dimension minus the number of cells of odd dimension.

Proposition 3.32. *Let X be a definable set. The Euler characteristic of $E_{\mathcal{D}}(X)$ does not depend on the particular cell decomposition $\mathcal{D}$ that we use to compute it.*

Proof. First we prove that if $X = C$ is a $(\varepsilon_1, \ldots, \varepsilon_m)$-cell, then for any cell decomposition $\mathcal{D}$ of X we have $E_{\mathcal{D}}(C) = E(C)$. We proceed by induction on m. For $m = 0$, it is clear. Now, we assume the statement for m and we prove it for

$m+1$. Suppose C is a $(\varepsilon_1, \dots, \varepsilon_m, 1)$-cell (the other case is similar). Then, for a cell $B \in \pi(\mathcal{D})$ the set of cell in $\mathcal{D}$ that projects to B is of the form

$$\Gamma(f_0), \dots, \Gamma(f_t), (f_0, f_1), \dots, (f_{t-1}, f_t),$$

for certain definable functions f_i. Hence, the cells that projects onto B contributes $t(-1)^d + (t+1)(-1)^{d+1} = -E(B)$ to $E_{\mathcal{D}}(C)$. Therefore we have

$$E_{\mathcal{D}}(C) = -\sum_{B \in \pi(\mathcal{D})} E(B) = -E_{\pi(\mathcal{D})}(\pi(C)) = E(C),$$

where in the second equality we have used the inductive hypothesis.

Now let X be any definable set and let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two different cell decomposition of X. Using the cell decomposition theorem we can find a cell decomposition $\mathcal{D}$ that is compatible with both $\mathcal{D}_1$ and $\mathcal{D}_2$. Notice that every cell in $\mathcal{D}_1$ is decomposed by cells in $\mathcal{D}$. Hence, using the previous paragraph, we can conclude $E_{\mathcal{D}_1}(X) = E_{\mathcal{D}}(X)$. Similarly, we also have $E_{\mathcal{D}_2}(X) = E_{\mathcal{D}}(X)$, and therefore $E_{\mathcal{D}_1}(X) = E_{\mathcal{D}_2}(X)$. □

In view of the Proposition 3.32, we can define the *Euler characteristic* of a definable set X as $E(X) = E_{\mathcal{D}}(X)$, where $\mathcal{D}$ is any cell decomposition of X.

Corollary 3.33. *Let X and Y be definable sets. Then $E(X \cup Y) = E(X) + E(Y) - E(X \cap Y)$.*

Proof. Let's consider first the case when X and Y are disjoint. Let $\mathcal{D} = \mathcal{D}_X \cup \mathcal{D}_Y$, where $\mathcal{D}_X$ and $\mathcal{D}_Y$ are cell decomposition of X and Y respectively. Then, $\mathcal{D}$ is a cell decomposition of $X \cup Y$ and clearly

$$E(X \cup Y) = E_{\mathcal{D}}(X \cup Y) = E_{\mathcal{D}_1}(X) + E_{\mathcal{D}_2}(Y) = E(X) + E(Y).$$

For the general case, notice that

$$X \cup Y = [X \setminus (X \cap Y)] \cup (X \cap Y) \cup [Y \setminus (X \cap Y)],$$

and that all the sets on the right side of the equality are disjoint. □

Proposition 3.34. *Let $S \subseteq M^{m+n}$ be a definable set and let $\pi : M^{m+n} \to M^m$ be the projection on the first m coordinates. Then $E(S_a)$ takes only finitely many values as a varies in $\pi(S)$. Moreover the set $E_n = \{a \in \pi(S) : E(S_a) = n\}$ is definable and*

$$E(S) = \sum_{n \in \mathbb{Z}} n \cdot E(E_n).$$

Proof. Consider a cell decomposition $\mathcal{D}$ of S and let $B \in \pi(\mathcal{D})$. Then, for any $b \in B$ its easy to see that $E(S_b)$ is constant. It follows that we can write the sets E_n, for $n \in \mathbb{Z}$ as a finite union of cells in $\pi(\mathcal{D})$. The formula follows easily from this. □

Corollary 3.35. *Let $S \subseteq M^{m+n}$ be a definable set and let $\pi : M^{m+n} \to M^m$ be the projection on the first m coordinates. Suppose that $E(S_a) = e$ is constant for any $a \in \pi(S)$. Then*

$$E(S) = E(\pi(S)) \cdot e.$$

In particular, if X, Y are definable sets, then $E(X \times Y) = E(X)E(Y)$.

Contrarily to the topological Euler characteristic $\chi(X)$ of a topological space X, the definable Euler characteristic is not invariant for (definable) homotopies. To see this, just notice that $E((0,1)) = 1$, $E([0,1)) = 0$, $E([0,1]) = -1$, even though all the intervals are homotopic. However, for the definable Euler characteristic we have the following fundamental invariance result.

Theorem 3.36 ([vdD98, Chapter 4, (2.4)])**.** *If $f : S \to M^n$ is a definable injection (not necessarily continuous), then*

$$E(S) = E(f(S)).$$

Sketch of the proof. By taking a cell decomposition, is not difficult to see that $E(S) = E(\Gamma(f))$. Similarly, we see that $E(f(S)) = E(\Gamma'(f))$, where $\Gamma'(f) = \{(f(x), x) : x \in S\}$ is the reversed graph of f. To conclude, it suffices to show that the Euler characteristic of a set is invariant for permutation of the coordinates. As every permutation can be written as a composition of transposition, we can restrict to the case where we exchange two coordinates. This last case is dealt with a lengthy double induction on the dimension of the space and on the dimension of the set. $\square$

Until now we have introduced two invariants for definable bijections, namely the dimension $\dim(X)$ of a definable set X and its Euler characteristic $E(X)$. Amazingly, if M extends a ordered field, then this is a complete set of invariants.

Fact 3.37 ([vdD98, Chapter 8, (2.11)])**.** *Let M be an o-minimal expansion of an ordered field and let $A \subseteq M^n$ and $B \subseteq M^m$ be two definable sets. Then*

$$A \text{ is in definable bijection with } B \iff \dim(A) = \dim(B) \text{ and } E(A) = E(B)$$

In general, the theorem fails even if M is an expansion of an ordered group, as there can be intervals having different cardinalities.

Chapter 4

O-minimal groups

4.1 Strzebonski's theory

Many important early results on the structure of definable groups, in particular on the torsion points, were obtained by Strzebonski in [Str94]. His approach to the study of definable groups can be seen in many ways as a generalisation of the study of finite groups, using Euler characteristic in place of the cardinality. This perhaps should not come as a surprise: the o-minimal Euler characteristic, contrarily to the topological one, satisfy many of the same properties of cardinality, for example it is additive and invariant by definable bijection. By virtue of this, classical theorems such as Sylows' Theorems and their proofs can be generalised without many difficulties, provided we use the right definitions and do some preparatory work. It should however be noted that, contrarily to cardinality, the Euler characteristic of a definable group can be zero, and the theory sholud also account for this case.

Definition 4.1. Let p be a prime. We say that G is a *p-group* if

$$E(G/H) \equiv 0 \mod p.$$

We say that G is a *strong p-group* if moreover $E(G) \neq 0$.

As it is the case for finite groups, the study of definable group actions plays a major role. Let S be a definable set and H a definable group. By a definable action of H on S we mean any definable map $\phi : H \times S \to S$ that is also a group action. As usual, for any $h \in H$ and $s \in S$ we will denote $\phi(h, s)$ simply as $h \cdot s$.

Lemma 4.2. *Let p be a prime. Let H be p-group that acts definably on a definable set S. Then, $E(S) \equiv E(S^H) \mod p$.*

Proof. Let R be the equivalence relation of being in the same H-orbit, i.e.

$$R = \{(x, y) \in S \times S : \exists h \in H \ y = hx\}.$$

By Proposition 3.34, we can write

$$E(S) = \sum_{n \in \mathbb{Z}} nE(\{x \in S/R : E(\mathrm{Orb}(x)) = n\}).$$

Using the bijection $\mathrm{Orb}(x) \cong H/\operatorname{Stab}(x)$, we have

$$E(S) = \sum_{n\in\mathbb{Z}} nE(\{x \in S/R : E(H/\operatorname{Stab}(x)) = n\}).$$

There are now two cases to consider: either $H = \operatorname{Stab}(x)$, hence $x \in S^H$, or $\operatorname{Stab}(x)$ is a proper subgroup of H and therefore $E(H/\operatorname{Stab}(x)) \equiv 0 \mod p$. By what we just said we can rewrite the sum as

$$E(S) = E(S^H) + p \sum_{n\equiv 0 \bmod p} (n/p)E(\{x \in S/R : E(H/\operatorname{Stab}(x)) = n\}).$$

From this, we easily conclude $E(S) \equiv E(S^H) \mod p$. □

Theorem 4.3 (Cauchy's Theorem)**.** *Let G be a definable group and let p be a prime. Then G has an element of order p if and only if $p \mid E(G)$.*

Proof. ($\Rightarrow$) Let $g \in G$ be such that $g^p = e$ and let K be the subgroup generated by g. Since K finite, it is definable and $E(K) = |K| = p$. To conclude, it suffice to notice that $E(G) = E(G/K)E(K)$.

($\Leftarrow$) Let $S \subseteq G\times\ldots\times G$ be the set of p-uples $(a_1, \ldots , a_n)$ such that $a_1\cdot\ldots\cdot a_n = e$. Notice that the value of a_n is determined by this condition and by the values of a_i, for $i < n$. Hence, it is easy to see that S is in definable bijection with G^{p-1}, and therefore $E(S) = E(G)^{p-1}$.

Now, consider the definable action of $H = \mathbb{Z}/p\mathbb{Z}$ on S given by $(a_1, \ldots , a_n) \mapsto (a_{1+h}, \ldots , ga_{n+h})$ for each $h \in H$. The tuples fixed by this action are precisely the ones in the form $(g, \ldots , g)$ where $g \in G$ and $g^p = e$. Therefore to conclude we only have to find a fixed tuple different from $(e, \ldots , e)$. By Lemma 4.2, we have $E(S^H) \equiv E(S) \equiv 0 \pmod p$. Hence, there must be at least a fixed tuple $(g, \ldots , g) \in S^H$ different from $(e, \ldots , e)$. □

Using this theorem, we obtain a characterization of definable groups with zero Euler characteristic: they are precisely the definable groups with torsion elements for any prime number. In the early years of the study of definable groups, one of the major topic was to prove that definably compact groups have torsion. This characterization effectively reduced the problem to showing that definably compact groups have zero Euler characteristic.

A way to prove this would be to use the theory of H-spaces, generalised to the o-minimal case by Edmundo and Otero in [EO04], and noticing that, by the structure theorem for Hopf-algebra they reference, any H-space has zero Euler characterisic. A solutions to this problem however used Thom's isomorphism to prove that the Euler class of a group (and therefore its Euler characteristic) is null. Precisely, Berarducci and Otero proved it in [BO03] transfering the problem on a real manifold and then using the classical version of Thom's isomorphism. A later proof by Edmundo and Woerheide do this directly on definable manifolds [EW09].

Going back to Strzebonki original theory, we now go toward a proof of the o-miniamal First Sylow Theorem.

Lemma 4.4. *Let G be a definable group and $H < G$ be a definable p-subgroup. Then, $E(G/H) \equiv E(N_G(H)/H) \mod p$.*

Proof. Consider the action of H on $S = G/H$ given by $xH \mapsto hxH$, for each $h \in H$. We claim that S^H is precisely $N_G(H)/H$, from which we can conclude using Lemma 4.2.

Hence, suppose $x \in S^H$. As $hxH = xH$ for every $h \in H$, we easily get $x^{-1}Hx \subseteq H$. Now, suppose that $x^{-1}Hx \subsetneq H$. Then, we would have an infinite descending chain $H \supsetneq x^{-1}Hx \supsetneq x^{-2}Hx^2 \supsetneq \dots$, contradicting the fact that, by Proposition 4.17, definable groups satisfy the descending chain condition on definable subgroups. □

Corollary 4.5. *Let G be a definable group and let $H < G$ be a definable p-subgroup. Suppose that $E(G/H) \equiv 0 \mod p$. Then $N_G(H) \neq H$.*

Proof. Follows immediately from the previous lemma. □

Theorem 4.6 (First Sylow Theorem)**.** *Let G be a definable group and let p be a prime. If $H < G$ is a (strong) p-subgroup of G and $E(G/H) \neq 0$, then there exists a (strong) p-subgroup $K < G$, such that H is normal in K.*

Proof. By Corollary 4.5, H is a proper normal subgroup of $N_G(H)$ so $N_G(H)/H$ is a nontrivial definable group. Moreover, by Lemma 4.4, we have

$$E(N_G(H)/H) \equiv E(G/H) \equiv 0 \mod p,$$

therefore $N_G(H)/H$ has a subgroup K/H of order p. We claim that K is a (strong) p-subgroup.

To prove that if H is strong then K is strong, it suffices to notice that $E(K) = E(K/H)E(H)$, so $E(H)$ is non-zero if and only if $E(K)$ is non-zero.

It remains to show that if M is a proper subgroup of K, then $E(K/M) \equiv 0 \mod p$. We consider two cases: $H \cap M = H$ or $H \cap M \neq H$. In the first case, we have $H < M$. Therefore, we can use the bijection $K/H \cong K/M \times M/H$ to conclude

$$E(K/M) = E(K/H)E(M/H) \equiv 0 \mod p.$$

For the second case, consider the subgroup $M < H \cdot M < K$. Using the bijection $K/M \cong K/(H \cdot M) \times (H \cdot M)/M$, we have

$$E(K/M) = E(K/(H \cdot M))E((H \cdot M)/M).$$

Now, using the bijection $(H \cdot M)/M \cong H/(H \cap M)$, we also have

$$E((H \cdot M)/M) = E(H/(H \cap M)).$$

As H is a p-subgroup, we can conclude $E(H/(H \cap M)) \equiv 0 \mod p$, and by the previous equality we also have $E(K/M) \equiv 0 \pmod p$. □

Corollary 4.7. *Strong p-groups are finite groups of order p^n, $n \geq 0$.*

Proof. Let G be a strong p-groups and let $E(G) = mp^n$, $p \nmid m$. By iterated application of the previous theorem, starting with $H = \{e\}$, we obtain a finite subgroup $K < G$ such that $E(K) = p^n$. If $K = G$ we are finished. Else, K is a proper subgroup of G, hence $m = E(G/K) \equiv 0 \mod p$. However, $p \nmid m$. □

The following corollary is one of the most useful consequences of Strzebonki's theory to the study of definable groups.

Corollary 4.8. *Let G be a definable abelian group and suppose $nG = 0$ for some $n > 0$. Then G is finite. In particular, the n-torsion subgroup of any definable abelian group is finite.*

Proof. First, notice that $E(G) \neq 0$. In fact, if it were zero, G would have element of each prime order, contradicting that the fact that the order of each element must divide n.

Now, let $E(G) = p_1^{m_1} \cdot \ldots \cdot p_k^{m_k}$. For each $i < k$ let H_i be a p_i-Sylow supgroup, and let $H = H_1 \cdot \ldots \cdot H_k$ be their product.

It is easy to see that $E(G/H) = 1$. Now, using the fact that G/H is a torsion group, and that the torsion of each element should divide $E(G/H)$, we can conclude that G/H is trivial and hence that $G = H$. To conclude recall that, as each H_i is a strong p_i-subgroups, by the previous corollary H is finite. □

Finally, we move to the Second Sylow Theorem. As in the finite case, we call *Sylow p-subgroups* the maximal p-subgroups of a definable group G. With this notation we have:

Theorem 4.9 (Second Sylow Theorem)**.** *Let H be a definable p-subgroup of a definable group G and let P be a Sylow p-subgroup of G. Then, there is $x \in G$ such that $H < xPx^{-1}$. In particular, any two Sylow p-subgroups are conjugate.*

Proof. Consider the action of H on G/P given by $gP \mapsto hgP$ for $h \in H$. It is easy to see that if $gP \in G/P$ is a fixed point of this action, then $H < gPg^{-1}$.

By Lemma 4.2, we have $E((G/P)^H) \equiv E(G/P) \pmod p$, and by the First Sylow Theorem $E(G/P) \not\equiv 0 \pmod p$. Therefore, a fixed point exists. □

Generalizing the Third Sylow Theorem to the o-minimal setting raises the problem of how to define the Euler characteristic of a definable family of sets. This was solved by Strzebonki by restricting to what he calls parametrizable families. The resulting theory, albeit very interesting, is rarely used in the study of definable groups, and we shall omit it. For an exposition of this theory, and the proof of the Third Sylow Theorem, we refer the reader to [Str94].

4.2 Definable Lie groups

Pillay proved quite early in [Pil88] that o-minimal groups can always be endowed with a unique topology, called the t-topology, making them topological groups. If the theory extends a field, the statement can be further strengthened: we can find a differential structure making them definable Lie groups. The following proof of the existence of the t-topology is from [BM13] and simplify the original proof. Recall that we say that a subset $X \subseteq G$ is large if $\dim(G \setminus X) < \dim(G)$.

Theorem 4.10. *Let $G \subseteq M^n$ be a definable group in an o-minimal structure M. Then G has a group topology, called the t-topology, which coincides with the topology induced by M^n on a large open subset V of G.*

Sketch of the proof. Let $Y \subseteq G \times G \times G$ be the set of points $(a, b, c) \in G \times G \times G$ such that $(x, y, z) \mapsto xyz \in G$ is continuous in a neighborhood of (a, b, c). By the cell decomposition theorem Y is large (and open) in G. Let V be the set of points $x \in G$ such that for most $(g_1, g_2) \in G \times G$ the triples (g_1, x, g_2) and $(g_1, g_1^{-1}xg_2^{-1}, g_2)$ belong to Y. Notice that V_0 is definable by definability of dimension. Moreover, V_0 contains all generic points of G, so it is large in G. Now, let V be the interior of V_0, so that V is definable, large and open. We define $O \subseteq G$ to be t-open if for all $a, b \in G$ the subset $aOb \subseteq V$ is open in V.

Notice that this amounts to say that we want to think of the sets in the form aVb as local charts of G. We have to verify that the change of chart is a continuous function:

Claim: For all $a, b \in G$, $Z = V \cap aVb$ is open in V , and the function $f : x \mapsto a^{-1}xb^{-1}$ from Z to V is continuous.

Using the previous claim is not difficult to prove that an subset of V open for the ambient topology is also open for the t-topology.

Now, we have to prove that the group operation and the inverse are t-continuous. Suppose for example we want to prove that the inverse is continuous at a point $b \in G$. Fix a generic point $a \in V$. Since a^{-1} is also generic, we have $a^{-1} \in V$. Since a is generic, the inverse must be continuous on a, and hence t-continuous since V is a chart. Now, notice this implies that the map $x \mapsto (xb^{-1}a)^{-1}$ is t-continuous in b. Finally, the map $x \mapsto x^{-1} = b^{-1}a(xb^{-1}a)$ is also continuous, as it is just the composition by a left translation. We reason similarly to prove that the group operation is t-continuous. □

We now introduce the concept of a definable C^p-manifold.

Definition 4.11. Let $p \geq 0$. If $p \neq 0$, we assume that M extends an ordered field.

A *definable chart* on X is a triple $\mathfrak{c} = (U, \phi, n)$, where U is a definable subset of X, $n > 0$ and ϕ is a definable bijection from U onto an open subset of M^n. We say that two definable charts $\mathfrak{c} = (U, \phi, n)$ and $\mathfrak{c}' = (U', \phi', n')$ are C^p-compatible if either $U \cap U' = \emptyset$ or $\phi(U \cap U')$ is open, $\phi'(U' capU)$ is open, and the two transition mappings $\phi \circ \phi'^{-1}$ and $\phi' \circ \phi^{-1}$ are of class C^p on their domains.

A *definable C^p-atlas* on X is a finite set $\mathfrak{C}$ of definable charts on X, each pair of which is C^p-compatible and whose domains cover X.

A *definable C^p-manifold* X is a pair $(X, \mathfrak{C})$, where X is a definable set and $\mathfrak{C}$ is a definable C^p-atlas on X.

Theorem 4.12. *If G is a group definable in M, then there is an atlas $\mathfrak{A}$ on G such that $(G, \mathfrak{A})$ is a definable C^p-group.*

Proof. For $p = 0$, this is a restatement of Theorem 4.10. For $p > 0$, the same proof works when one replaces "continuous" by "C^p". □

A definable group G has a unique differential structure compatible with the group operation, as we now shall prove.

Proposition 4.13. *Let M be an o-minimal structure that extends a field and let $(G, \mathcal{A})$ and $(H, \mathcal{B})$ be definable C^k-groups. Let $f : G \to H$ be a definable group homomorphism, then f is C^k relatively to the manifold structure.*

Proof. By the cell decomposition theorem we can find a local chart $A \in \mathcal{A}$, a point $x_0 \in A$ and a local chart $B \in \mathcal{B}$ such that $f(x_0) \in B$ and $f : A \to B$ is C^k in x_0. Now, fix any point $g_0 \in G$ and let $h \in G$ be such that $hg_0 = x_0$. We can write $f(g) = f(h)^{-1}f(hg)$. Since the group operation is C^k in both G and H, this proves that f is continuous in g_0. □

Corollary 4.14. *If G is a definable group, the definable C^p-group structure on G is unique.*

Proof. Let $(G, \mathcal{A})$ and $(G, \mathcal{B})$ be two differential structures on G. Applying the previous proposition to the identity $\mathrm{id} : (G, \mathcal{A}) \to (G, \mathcal{B})$, we obtain a diffeomorphism between them. □

Lastly, we close this section by establishing some basic facts on definable groups and by proving that they satisfy the descending chain condition on definable subgroups.

Proposition 4.15. *For a definable subgroup H of G the following conditions are equivalent.*

1. *H has a finite index in G.*
2. *$\dim(H) = \dim(G)$.*
3. *H contains an open neighborhood of e.*
4. *H is open in G.*

Proof. (4) $\Rightarrow$ (3). Clear.

(3) $\Rightarrow$ (2). Work in a chart containing e. If H contains an open neighborhood of the identity, then by definition it contains a small cube of containing e and of dimension equal to the dimension of the chart. Therefore H itself has the same dimension of the chart, and therefore the same dimension of G.

(2) $\Rightarrow$ (4). First, notice that there must be an chart A of G such that $\dim(H \cap A) = \dim(H)$. Work in this chart. There must be a cell contained in H of dimension equal to $\dim(H)$. As it is the same of the dimension of the chart, the interior of H in A is non-empty. Therefore, being a topological subgroup, H is open.

(4) $\Rightarrow$ (1). As H is open, H is also closed. The number of clopen subsets of G is bounded by the number of connected components of G, and therefore it is finite.

(1) $\Rightarrow$ (2). If H has finite index, then $G = g_1 H \cup \ldots \cup g_n H$. Therefore, $\dim(G) = \max_{i \leq n} \dim(g_i H) = \dim(H)$. □

Proposition 4.16. *Let G be a definable group and $H < G$ a definable subgroup. Then H is closed in the t-topology.*

Proof. Consider the closure $\bar{H}$, it is still a topological group and H is a subgroup with $\dim(H) = \dim(\bar{H})$. By Proposition 4.15, H is open in $\bar{H}$ and therefore also closed. Hence, $\bar{H} = H$. □

Proposition 4.17. *Let G be a definable group. Then G satisfies the descending chain condition on definable subgroups.*

Proof. Let $H_1 > H_2 > \ldots$ be an infinite descending chain of definable subgroups. By Proposition 4.15, either $\dim H_{i+1} < \dim H_i$ or index of H_{i+1} in H_i is finite. Hence, after a finite number of steps, the chain must stabilize. □

4.3 Lie algebras

Having proved the existence of a differential structure on definable groups, in this section we show how to give a Lie algebra structure to the tangent at the identity of the group. We will then establish a connection between the Lie algebra structure and the group structure.

Many of the facts contained in this section are generalizations of classical facts on Lie groups, and their proofs are only sightly different in the definable case. For this reason, we will only state them, without proofs, for future reference. We refer the reader to [PPS00] for full details.

Definition 4.18. A vector space L over k with a bilinear operation $[\ ,\ ] : L \times L \to L$ is a Lie algebra over k if $[x, x] = 0$ for all $x \subseteq L$ and $[\ ,\ ]$ satisfies the Jacobi identity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Definition 4.19. We use the following classical definitions:

- A subspace I of L is called an ideal of L if $[x, y] \subseteq I$ for all $x \in I$, $y \subseteq L$.
- An ideal I of L is abelian if $[x, y] = 0$ for all $x, y \in I$.
- A Lie algebra L is simple if it has no ideals except itself and $\{0\}$.
- A Lie algebra is semisimple if it has no abelian ideals except for $\{0\}$.

Definition 4.20. Let G be a definable group. We say that:

- G is definably simple if G is nonabelian and has no proper definable nontrivial normal subgroup
- G is semisimple if all its abelian normal subgroups are discrete.

Now, let G be a definable group and let $\Psi : G \to \mathrm{Aut}(G)$ be the conjugation map $\Psi_g(x) = gxg^{-1}$. We define the *adjoint map* $\mathrm{Ad} : G \to \mathrm{Aut}(\mathcal{T}_e(G))$ by

$$\mathrm{Ad}(g) = \mathrm{d}_e\Psi_g.$$

Notice that Ad is a group homomorphism.

Finally, let $\mathcal{T}_e(G)$ be the tangent space at the identity of G and let $\mathrm{ad} : \mathcal{T}_e(G) \to \mathrm{End}(\mathcal{T}_e(G))$ be given by $\mathrm{ad} = \mathrm{d}_e\,\mathrm{Ad}$. We can use this map to define a Lie algebra structure on the tangent space of the group.

Fact 4.21 ([PPS00, 2.27])**.** *Define a bilinear form on $\mathcal{T}_e(G)$ by letting $[v, w] = \mathrm{ad}(v)(w)$. Then $\mathfrak{g} = (\mathcal{T}_e(G), [\ ,\ ])$ is a Lie algebra that we call the* Lie algebra of the group*.*

One very useful feature of the adjoint map is that it gives a way to embed a centerless group inside the authomorphism group of its Lie algebra. To see this, we first need the following fact that is proved in [OPP96] by exhibiting a criterion for the uniqueness of solutions to differential equations in o-minimal structures.

Fact 4.22 ([OPP96, Lemma 3.2])**.** *Let $\langle G, e, \cdot\rangle$ be a definably connected group. Let σ, τ be M-definable group homomorphisms of G. Then*

(i) the maps σ, τ are differentiable everywhere on G;

(ii) if $\sigma \in \tau$, then $\mathrm{d}_e(\sigma) \neq \mathrm{d}_e(\tau)$.

Fact 4.23. *Let G be a definable group and let $g \in G$. Then $\mathrm{Ad}(g)$ is an authomorphism of the Lie algebra $\mathfrak{g}$. Therefore, we have a group homomorphism $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$.*

Lemma 4.24. *Let G be a definable group. Then, the kernel of the adjoint map is $Z(G)$.*

Proof. Clearly $Z(G) \subseteq \ker \mathrm{Ad}$. Conversely, suppose that $\mathrm{Ad}(g) = \mathrm{d}_e\Psi_g = \mathrm{id}$ for some $g \in G$. Then, Fact 4.22 tells us that $\Psi_g = \mathrm{id}$, so $g \in Z(G)$. □

Finally, we list some facts about Lie algebra and their connection with definable Lie groups.

Fact 4.25 ([PPS00, 2.8])**.** *If $L = (V, [\ ,\ ])$ is a semisimple Lie algebra over a real closed field $\mathcal{R}$, where V is a subspace of $\mathcal{R}^n$, then $\dim(L) = \dim(\mathrm{Aut}(L))$.*

Fact 4.26 ([PPS00, 2.35])**.** *Let G be a definably connected definable semisimple group with Lie algebra $\mathfrak{g}$. If $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, then there is a definably connected definable normal subgroup H of G whose Lie algebra is $\mathfrak{h}$.*

Fact 4.27 ([PPS00, 2.36])**.** *Let G be a definably connected centerless definable group. G is definably simple if and only if its Lie algebra $\mathfrak{g}$ is simple.*

4.4 Structure theorems for definable groups

We now present some useful structure theorems from [PPS00]. Using them, we can often reduce the study of o-minimal groups to the study of definable abelian groups and definably simple groups. Notice that several more advanced structure theorems have been proved in the following years (cf. [HPP11]), however, for our work, these will suffice.

Proposition 4.28. *A definably simple group is definably isomorphic to a semialgebraic subgroup of $GL(n, M)$ defined over $\mathbb{R}_{alg} \subseteq M$.*

Proof. Since G is simple, Lemma 4.24 tells us that $\ker \mathrm{Ad} = 1$, hence Ad is an embedding of G in $\mathrm{Aut}(\mathfrak{g})$. Let $H = \mathrm{Ad}(G) < \mathrm{Aut}(\mathfrak{g})$. Looking at dimensions, by Fact 4.25, we have

$$\dim(\mathrm{Aut}(\mathfrak{g})) = \dim(G) = \dim(H),$$

thus H is a subgroup of $\mathrm{Aut}(\mathfrak{g})$ of maximal dimension. Hence, H is the disjoint union of finitely many translates of the connected component $\mathrm{Aut}(\mathfrak{g})^0$. Since $\mathrm{Aut}(\mathfrak{g})$ is semialgebraic, it is not difficult to see that $\mathrm{Aut}(\mathfrak{g})^0$ too is semialgebraic, and therefore so is H.

Now we improve the previous result by showing that $\mathfrak{g}$ is isomorphic to a Lie subalgebra $L < \mathfrak{gl}(n, M)$ defined over $\mathbb{R}_{alg}$. It then follows, reasoning as before, that G is in fact isomorphic to the semialgebraic group $\mathrm{Aut}(L)$ defined over $\mathbb{R}_{alg}$.

Recall that, by the classification theorem for simple Lie algebras, there are finitly many Lie subalgebras $L_1, \ldots, L_n < \mathfrak{gl}(n, \mathbb{R})$ such any Lie subalgebra of $\mathfrak{gl}(n, \mathbb{R})$ is isomorphic to L_i for some i. Notice that using generators, we can write this as a first order statement in $(\mathbb{R}, <, +, \cdot)$. By transfer, this statement also holds in $\mathbb{R}_{alg}$, hence we can take the L_i to be defined over $\mathbb{R}_{alg}$. Since M too is an RCF, the statement is also true in M. It follows that $\mathfrak{g}$ is definably isomorphic to L_i for some i. To conclude, take $L = L_i$. $\square$

Lemma 4.29. *Let G be a definably connected semisimple group. Then $G/Z(G)$ is centerless.*

Proof. First, notice that, since G is semisimple, $Z(G)$ is finite. Suppose that there is some $g \in G$ such that $gZ(G) \in Z(G/Z(G))$. This means that the continuous map $G \ni x \mapsto gxg^{-1}x^{-1}$ maps G in $Z(G)$. As $Z(G)$ is discrete and G is definably connected, we must have $gxg^{-1}x^{-1} = e$ for each $x \in G$, thus $g \in Z(G)$. $\square$

Theorem 4.30. *Let G be a definably connected semisimple group. Then $Z(G)$ is finite and $G/Z(G)$ is definably isomorphic to a direct product of semialgebraic subgroups of $GL(n, M)$ defined over $\mathbb{R}_{alg}$.*

Proof. By the previous lemma, $G/Z(G)$ is centerless. Therefore, it suffices to prove the theorem for G a definably connected semisimple centerless group.

Let $\mathfrak{g}$ be the Lie algebra of G. Since G is semisimple, $\mathfrak{g}$ is also semisimple. Therefore, $\mathfrak{g}$ can be written as the direct sum of ideals $\mathfrak{h}_1, \ldots, \mathfrak{h}_n < \mathfrak{g}$ such that each ideal is a simple Lie algebra. For each $i < n$, let $H_i < G$ be the normal subgroup corresponding to $\mathfrak{h}_i$. Notice that, as $\mathfrak{h}_i$ is a simple Lie algebra, H_i is simple. We want to show that G is the direct product of $H_1, \ldots, H_n$.

As $[\mathfrak{h}_i, \mathfrak{h}_j] = 0$ for $i \neq j$, H_i commutes with H_j. In particular, $H_i \cap H_h$ is normal in H_i. As H_i is simple, we have $H_i \cap H_j = \{e\}$.

Hence, $H = H_1 \cdot \ldots \cdot H_n$ is a direct product and its dimension is

$$\dim(H) = \dim(H_1) + \ldots + \dim(H_n).$$

On the other hand we have $\dim(H_i) = \dim(\mathfrak{h}_i)$ and

$$\dim(G) = \dim(\mathfrak{h}_1) + \ldots + \dim(\mathfrak{h}_n),$$

so $\dim(H) = \dim(G)$. As G is definably connected, it follows that $H = G$. $\square$

4.5 Descending chain condition

In this section we prove that definable groups in o-minimal theories satisfy the descending chain condition (dcc) on type-definable subgroups of bounded index. This topic was first considered after Pillay noted in [Pil04], that the dcc is equivalent to G/G^{00} being a compact Lie group. In fact, the equivalence follows easily from the following classical theorem on compact groups and by the definition of logic topology.

Fact 4.31 ([HM13])**.** *A compact group is a Lie group if and only if it has the descending chain condition for closed subgroups.*

A proof of the dcc on type-definable subgroups of bounded index was found later by the joint work of Berarducci, Otero, Peterzil, and Pillay [BOPP05]. The proof, that we now present, relies on the previously established structure theorems for definable groups to reduce to the semisimple and the abelian case. The semisimple case is comparatively easy to prove, owning to Theorem 4.30. For the abelian case instead, we will make use of the following classical theorem.

Fact 4.32 ([HM13, Theorem 8.36(i)])**.** *A compact connected locally connected abelian group whose topology has a countable basis is isomorphic to a (possibly infinite) torus.*

To be able to use this theorem, we first have to prove that G/G^{00}, endowed with the logic topology, is a locally connected space. We start by recalling some basic facts and definitions about locally connected spaces.

Definition 4.33. Let X be a topological space.

(i) We say that X is *locally connected* at a point $x \in X$ if for every open neighborhood U of x there is a connected open neighborhood V of x contained in U. The space X is said to be locally connected if it is locally connected at every point.

(ii) We say that X is *weakly locally connected* (or *connected im kleinen*) at a point $x \in X$ if for every open neighborhood U of x there is a (not necessarily open) connected neighborhood V of x contained in U. The space X is said to be weakly locally connected if it is weakly locally connected at every point.

Lemma 4.34. *A topological space X is locally connected if and only if it is weakly locally connected.*

Proof. First, notice that if for each open set U, the connected components of U are open, then the space admits a basis of connected open sets, and therefore it is locally connected.

Hence, we only need to prove that for any open set U and for any connected component C of U, C is open. Let $x \in C$ be a point. By definition of weakly locally connectedness, we can find a connected neighborhood V of x contained in U. Since V is connected and C is a connected component, we must have $V \subseteq C$. Therefore, C contains an open neighborhood of x. As this is true for any $x \in C$, we conclude that C is open. □

Lemma 4.35. *Let $V \subseteq G$ be a definably connected set and let $\pi : G \to G/G^{00}$ be the quotient projection. Then $\pi(V)$ is connected in the logic topology.*

Proof. Suppose that there are two disjoint closed sets $Z_1, Z_2 \subseteq G/G^{00}$ such that $\pi(V) = Z_1 \cup Z_2$. Let $V_1 = V \cap \pi^{-1}(Z_1)$ and $V_2 = V \cap \pi^{-1}(Z_2)$. By definition of logic topology they are type-definable and disjoint. Moreover, $V = V_1 \cup V_2$, hence by compactness V_1 and V_2 must be definable. Since $\pi^{-1}(Z_1)$ and $\pi^{-1}(Z_2)$ are open in G, V_1 and V_2 are also open in V, contradicting the fact that V is definably connected. □

Lemma 4.36. *Let $Y \subseteq G$ be a definable set containing G^{00}. Then $eG^{00} \in G/G^{00}$ is contained in the interior of $\pi(Y)$.*

Proof. Consider Y^c the complementary of Y. Clearly it is definable, so by definition of logic topology $\pi(Y^c)$ is a closed subset of G/G^{00} that does not contain eG^{00}. As $\pi(Y^c) \subseteq \pi(Y)^c$, this implies that eG^{00} is in the interior of $\pi(Y)$. $\square$

Lemma 4.37. *Let $Y \subseteq G$ be a definable set containing G^{00} and let Y^0 be the connected component of Y containing $e \in G$. Then $G^{00} \subseteq Y^0$.*

Proof. Using compactness, starting with $Y_0 = Y$ we define a sequence of subsets $Y_n \subseteq G$ such that:

1. $G^{00} \subseteq Y_n$
2. $Y_n = Y_n^{-1}$
3. $Y_{n+1}^2 \subseteq Y_n$

Notice that, since each Y_n contains G^{00}, they all have bounded index (meaning boundedly many translates cover G). Let Y_n^0 be the connected component of Y_n containing the identity $e \in G$, we claim that Y_n^0 has bounded index.

First notice that, as Y_{n+1} has bounded index, at least one of its connected component must have bounded index. Call one such component Z_{n+1} and choose $z \in Z_{n+1}$. Clearly, $z^{-1}Z_{n+1}$ has bounded index. Moreover, since $Z_{n+1}^2 \subseteq Y_{n+1}^2 \subseteq Y_n$, we have $z^{-1}Z_{n+1} \subseteq Y_n$ Using that Z_{n+1} is connected, we can conclude that $z^{-1}Z_{n+1} \subseteq Y_n^0$. Therefore Y_n^0 has bounded index.

Now, let $Y_\omega^0 = \bigcap_{n<\omega} Y_n^0$. Clearly, Y_ω^0 is a subgroup of G. Moreover, Y_ω^0 has bounded index in G because it is a decreasing intersection of countably many sets of bounded index. Hence, we have $G^{00} \subseteq Y_\omega^0 \subseteq Y^0$. $\square$

Proposition 4.38. *The compact group G/G^{00} is locally connected.*

Proof. Suppose that we manage to show that G/G^{00} is weakly locally connected at the identity $eG^{00} \in G/G^{00}$. Then we would have that G/G^{00} is weakly locally connected at every point, as translation, say to the right, is an homemorphism. Therefore, by Lemma 4.34 the space would be locally connected.

Hence we only have to prove that G/G^{00} is weakly locally connected at $e \in G/G^{00}$. Let $\pi : G \to G/G^{00}$ be the quotient map and let $U \subseteq G/G^{00}$ be an open neighborhood of the identity eG^{00}. We have to find a connected neighborhood V of eG^{00} contained in U. First, notice that the preimage $\pi^{-1}(U)$ is an $\bigvee$-definable set containing the type-definable set G^{00}. Thus, by compactness there is a definable set $Y \subseteq G$ such that $G^{00} \subseteq Y \subseteq \pi^{-1}(U)$. By Lemma 4.37, we may assume Y to be definably connected. We can conclude by putting $V = \pi(Y)$. In fact, by Lemma 4.35, V is connected and, by Lemma 4.36, $eG^{00} \in \mathrm{int}(V)$. $\square$

Lemma 4.39. *Let G be definably connected. Then, G^{00} is divisible.*

Proof. Let n be a positive integer. By Corollary 4.8, the ker of the map $n : G \to G$, given by $x \mapsto nx$, is finite. Therefore, the image is open, and hence, by definably connectedness we can conclude that $nG = G$. Now, nG^{00} is a type definable subgroup of bounded index in $nG = G$, therefore $G^{00} \subseteq nG^{00}$. On the other hand, $nG^{00} \subseteq G^{00}$. Therefore, $nG^{00} = G^{00}$, as we wanted. $\square$

Proposition 4.40. *Let G be a definable abelian group, then G has the dcc on type-definable subgroups of bounded index, i.e. there are no infinite descending chains $H_0 > H_1 > \dots$ of type-definable subgroups of bounded index.*

Proof. By Lemma 2.7 we can assume that each H_i is defined by only countably many formulas. Therefore, we can find a countable model M_0 and a countable language $L_0 \subseteq L$ over which all the H_i are defined. Let $H = G^{00}_{L_0}$ be the smallest bounded index subgroup type-definable in the language L_0 with parameters from M_0.

By definition of logic topology, G/H is a compact connected locally connected abelian group with a countable basis. Then, Fact 4.32 tells us that G/H is isomorphic to a (possibly infinite) torus. On the other hand, by Corollary 4.8 we know that G has finite 2-torsion and by Lemma 4.39, we know that H is divisible. Hence, G/H too must have finite 2-torsion. The only possibility is then for G/H to be isomorphic to a finite dimensional torus.

To conclude consider the chain $H_0/H > H_1/H > \dots$ of closed subgroups in G/H. By what we just said G/H is a Lie group, and by Fact 4.31 Lie groups have the dcc on closed subgroups. Therefore the chain must stabilize and the same must happen to the original chain $H_0 > H_1 > \dots$ in G. $\square$

Corollary 4.41. *Let G be a definable abelian group and suppose $H < G$ is a type-definable torsion-free subgroup of bounded index. Then $H = G^{00}$.*

Proof. Suppose $H \neq G^{00}$. By definition, $G^{00} \subseteq H$. Hence, H/G^{00} is a compact non-trivial abelian Lie group, and therefore it has torsion. However, G^{00} is divisible and H is torsion-free by hypothesis, hence H/G^{00} is torsion-free. Contradiction. $\square$

Proposition 4.42. *Assume that $G(\mathbb{R})$ is a simple semialgebraic compact Lie group defined over $\mathcal{M}_0 = (\mathbb{R}, <, +, \cdot)$. Then the standard part map $\mathrm{st} : G \to G(\mathbb{R})$ is a group homomorphism. Moreover, the infinitesimal neighborhood of the identity $\mu(e) = \ker(\mathrm{st})$ coincides with G^{00}, hence $G/G^{00} \cong G(\mathbb{R})$.*

Proof. Let $\mathcal{M}$ be a saturated extension of $\mathcal{M}_0$ and let T be a maximal torus of $G(\mathbb{R})$. Then T is a maximal abelian subgroup of G, and therefore it is definable in M_0 (by dcc on definable groups or, more generally, because the theory is NIP). Let $T(\mathcal{M})$ be the interpretation of the definition of T in $\mathcal{M}$. By classic Lie theory, the conjugates of T covers $G(\mathbb{R})$. As this is expressible as a first order statement, we also have that the conjugates of $T(\mathcal{M})$ cover G.

Since $\mu(e) \cap T(\mathcal{M})$ is torsion free, by Corollary 4.41, we have $\mu(e) \cap T(\mathcal{M}) = T^{00}(M)$. Now, let $H < G$ be a type-definable subgroup of bounded index and let T' be any conjugate of T. Then, $H \cap T'$ has bounded index, and therefore $\mu(e) \cap T'(\mathcal{M}) \subseteq H$. As this is true for any conjugate of T, and they cover G, we can conclude $\mu(e) \subseteq H$. Since $\mu(e)$ is a type-definable subgroup of bounded index and it is contained in every other type-definable subgroup of bounded index, we must have $G^{00} = \mu(e)$. $\square$

Lemma 4.43. *Let $N < G$ be a normal subgroup. Suppose both N and G/N have the dcc on type-definable subgroups of bounded index. Then G also have the dcc.*

Proof. Let $H_1 < H_2$ be type definable subgroups of bounded index, and suppose that H_1 is a proper subgroup of H_2. Then, $H_i \cap N$ and $H_iN//N$ are type definable subgroups of bounded index of N and G/N respectively. Moreover, either $H_i \cap N$ or H_iN/N must be proper. This is enough. □

Theorem 4.44. *Let G be a definable group in an o-minimal theory. Then G satisfy the descending chain condition on type-definable subgroups of bounded index.*

Proof. We proceed by induction on $\dim(G)$ using the decomposition theorems. We have the following cases:

G **definably simple.** By Proposition 4.28, we can assume that G is a semialgebraic group defined over $\mathbb{R}_{alg}$. We can therefore conclude using Proposition 4.42.

G **definably semisimple.** By Lemma 4.56, it suffices to prove this for $Z(G)$ and for $G/Z(G)$. The former is an abelian group, hence we can conclude using Proposition 4.40. By Theorem 4.30, $G/Z(G)$ is the direct product of definably simple groups. Thus, we can easily conclude using the previous case.

General case. If G is not a definably connected semisimple group, then G has an infinite normal commutative subgroup N. By Proposition 4.40, we know the theorem to be true for N. Thus, we may assume $G \neq N$. As $\dim(G/N) < \dim(G)$, by inductive hypothesis we know that G/N satisfy the dcc. As both N and G/N satisfy the dcc, by Lemma 4.43, G also satisfy the dcc. □

Corollary 4.45. *Let G be a definable group in an o-minimal theory. Then G/G^{00} is a compact Lie group.*

Proof. By Fact 4.31 it suffices to show that there are no infinite descending chains of closed subgroups. By contradiction, suppose that there is an infinite descending chain of $H_0 > H_1 > \ldots$ of closed subgroups. Then, by definition of logic topology, their preimages in G forms an infinite descending chain of type definable subgroups. By Theorem 4.44 such a chain must stabilize. □

4.6 Generically stable measures in o-minimal theories

In this section we prove that generically stable measures in o-minimal theories are always smooth. In particular, as we will better see in the next section, this implies that fsg groups in o-minimal theories are compactly dominated. The fact that generically stable measures are smooth was first proved for o-minimal theories by Simon in [Sim10]. Notice that the same result holds in a wider class of theories, the *distal theories* (cf. [Sim14]). However, we won't prove it in that generality.

Lemma 4.46. *Let $M \prec N$ be models of an o-minimal theory T. Let $\phi(x) \in L(N)$ be a formula in one variable. Let $a_1, \ldots, a_n$ be the endpoints of the intervals that constitute the set $\phi(x)$ such that $a_i \in N \setminus M$. Then $\partial^M \phi(x)$ is precisely* $\{\mathrm{tp}(a_1/M), \ldots, \mathrm{tp}(a_k/M)\}$.

Proof. If $a \in N$ is not an endpoint of $\phi(x)$, then let $b, c \in M$ be such that the interval (b, c) contains a and it is completely contained in $\phi(x)$ or in $\neg\phi(x)$. Then, $\mathrm{tp}(a/M)$ contains the formula $b < x < c$ and therefore it is not in the border. If, on the other hand, $a \in N \setminus M$ is an endpoint then both any realisation of $\mathrm{tp}(b^{\pm}/N)$ has the same type on M as b. One of those realisations satisfy $\phi(x)$, the other does not. $\square$

Lemma 4.47. *Let T be an o-minimal theory. Assume that every generically stable measure in one variable over $\mathcal{U}$ is smooth. Then any generically stable measure is smooth.*

Proof. Let $\mu \in \mathfrak{M}(M)$ be an M-invariant generically stable measure in k variables. We want to show that for any $N \supset M$ there is a unique extension of μ over N. For this it suffices to show that for any element $c \in \mathcal{U}$ there is a unique extension of μ to $M(c)$, in fact we can iterate this process with $M(c)$ in place of M until we reach N. Notice that any formula in $\mathcal{L}(M(c))$ can be written as $\phi(\bar{x}, c)$ with $\phi(\bar{x}, y) \in L(M)$. By the cell decomposition theorem, $\phi(\bar{x}, y)$ is a boolean combination of formula of the type $f(\bar{x}) < y$, where f is a M-definable function. So we can assume $\phi(\bar{x}, c) \equiv f(\bar{x}) < c$. Let ν be an any extension of μ to $M(c)$. Then $\nu(\phi(\bar{x}, c)) = \nu(f(\bar{x}) < c) = f_*(\nu)(z < c)$. Notice that $f_*(\mu)$ is a generically stable measure in one variable, so it is smooth by hypothesis. As $f_*(\nu)$ is an extension of $f_*(\mu)$ its value is determined, and so is also the value of $\nu(\phi(\bar{x}, c))$. $\square$

Theorem 4.48. *If T is o-minimal, then any generically stable measure is smooth.*

Proof. By the previous lemma, we only have to prove the theorem for measures in one variable. Then, by Lemma 1.30, it sufficies to show that the border of any formula in one variable has measure zero. Let $\phi(x, y) \in L(M)$ with $|x| = 1$ and $b \in \mathcal{U}$. Then by Lemma 4.46, $\partial^M \phi(x, b)$ is a finite set $\{\mathrm{tp}(a_1/M), \ldots, \mathrm{tp}(a_n/M)\}$. Suppose $\mu(\partial^M \phi(x, b)) > 0$, then at least one of these types a_i has positive μ-measure. The fact that μ is generically stable implies that a_i is generically stable. In fact if a type p has positive μ-measure then every formula it contains must have positive μ-measure so it must be finitely satisfiable because μ is. Moreover it is definable. By the previous discussion, a_i must also be unrealised. However an o-minimal theory does not have any non-trivial generically stable type. In fact, if p were a generically stable type, we should have $p_x \otimes p_y = p_y \otimes p_x$. However, this is impossible since one formula between $x < y$ and $y < x$ must be contained in $p_x \otimes p_y$ and the other would be contained in $p_y \otimes p_x$. Hence, $\mu(\partial^M \phi(x, b))$ must be zero. $\square$

4.7 Fsg groups in o-minimal theories

In this section we prove that definably compact groups in an o-minimal theory are fsg groups. From this, we can deduce one of the main results we present, i.e. that definably compact group are compactly dominated. The first proof that definably compact groups are fsg was given in [HPP08, Theorem 8.1]. The proof we present uses the same main ideas, however we concentrate on the existence of measures. Moreover, we give a different and shorter proof of some of the main lemmas. As in Section 4.5, we will make use of the decomposition theorems for o-minimal groups to reduce to the semisimple case and to the abelian case.

It may be useful to recall all the equivalent conditions that we have proved until now for definable groups in o-minimal theories. In the following, we will often use these equivalences implicitly.

Theorem. *Let G be a definable group in an o-minimal theory. Then, the followings are equivalent:*

(i) G is an fsg group.

(ii) G has a finitely satisfiable global invariant measure.

(iii) G has a global generically stable measure.

(iv) G has a smooth global invariant measure.

(v) G is compactly dominated as a group by G/G^{00}.

Proof. (i) $\Leftrightarrow$ (ii) By Proposition 2.20.

(ii) $\Leftrightarrow$ (iii) By Proposition 2.21.

(iii) $\Leftarrow$ (iv) By Lemma 2.27.

(iii) $\Rightarrow$ (iv) By Theorem 4.48. Notice that this is the only part that uses o-minimality.

(iv) $\Leftrightarrow$ (v) By Theorem 2.38. □

An important role will be played by the following characterization of forking in o-minimal theories. The statement we present below is from [PP07], however the original proof is due to Dolich [Dol04].

Fact 4.49 ([PP07, Theorem 2.1])**.** *Suppose X is a definable closed and bounded subset of M^n, and M_0 is a model (small elementary substructure of M). Then the following are equivalent:*

(i) X does not fork over M_0.

(ii) X has a point in M_0.

First, we prove that definably compact abelian group are fsg. The general case will then follow easily using the decomposition theorems.

Lemma 4.50. *Let $Y \subseteq G$ be a generic set. Then it contains a closed subset $Y' \subseteq Y$ that is still generic.*

Lemma 4.51. *Let G be a definable group and let $X \subseteq G$ be a definable subset whose closure in G is definably compact. If X is not left-generic then $G \setminus X$ is right-generic.*

Proof. Let M_0 be such that both X and G are defined over M_0. By Lemma 4.50 we may assume X to be closed. Suppose that G is not left generic. Then, for every $h_1, \dots, h_k \in G$ there is $g \in G$ such that $h_i \notin Xg$, for $i = 1, \dots, k$. By compactness, there is $g \in G$ such that Xg has no point in M_0.

Since the set Xg does not contain any point in M_0, by Fact 4.49 we have that Xg forks over M_0. This means that there are elemens $g_1, \dots, g_n \in G$ realizing $\mathrm{tp}(g/M_0)$ and such that $Xg_1 \cap \ldots \cap Xg_n = \emptyset$. Passing to the complementary we obtian our statement. □

Corollary 4.52. *In a definably compact abelian group G, the non-generic sets form an ideal. In particular, definably compact abelian groups have generic types.*

Proof. First, notice that in an abelian group right-generics coincide with left-generics. Now, let X_1 and X_2 be two non-generic such that $X = X_1 \cup X_2$ is generic. Then there are $g_1, \dots, g_n \in G$ such that

$$G = \bigcup_{i \leq n} g_i X_1 \cup \bigcup_{i \leq n} g_i X_2.$$

Suppose $\bigcup_{i \leq n} g_i X_2$ is not generic. Then, by previous lemma, $\bigcup_{i \leq n} g_i X_1$ is generic, and this implies that X_1 is generic.

For the second part it is clearly enough to show that, given a partial type $\Sigma(x)$ such that every finite conjunction of formulas in $\Sigma(x)$ defines a generic set, and given a formula $\phi(x, b) \in L(\mathcal{U})$, at least one partial type between $\Sigma(x) \cup \{\phi(x, b)\}$ and $\Sigma(x) \cup \{\neg\phi(x, b)\}$ has the same property. Hence, suppose by contradiction that there are formulas $\psi_1^\varepsilon(x, b_1^0), \dots, \psi_n^\varepsilon(x, b_n^0)$ in $\Sigma(x)$, for $\varepsilon \in \{0, 1\}$ such that both $\psi_1^0(x, b_1^0) \wedge \dots \wedge \psi_n^0(x, b_n^0) \wedge \phi(x, b)$ and $\psi_1^1(x, b_1^1) \wedge \dots \wedge \psi_n^1(x, b_n^1) \wedge \neg\phi(x, b)$ are non-generic. Then, by the previous part, their disjunction is non-generic. However their disjunction is a finite conjunction of formualas in $\Sigma(x)$. Contradiction. □

Lemma 4.53. *Let G be a definably amenable group in a NIP theory. Then, there is a model N over which generic sets do not fork.*

Proof. Since G is definably amenable, by Lemma 2.17, it has a global invariant measure μ definable over some small model N. In particular, μ is N-invariant and hence, by Corollary 1.19, it does not fork over N. Now, let X be a generic set. As a finite number of translates of X covers G, X must have positive μ-measure. Hence, since μ is non-forking, X does not fork. □

Proposition 4.54. *Definably compact abelian groups are fsg.*

Proof. Given, by Corollary 4.52, the existence of a generic type, it is enough to prove that all generic sets are finitely satisfiable over some small model N. Notice that, since G is abelian, it is also amenable. We can then apply Lemma 4.53 to find a small model N over which generic sets does not fork. Now, let X be a generic set. By Lemma 4.50, we may assume X to be closed. Since X is generic, by our choice of N we have that X does not fork over N. Hence, by Fact 4.49, X is satisfiable in N. □

Now that we have finished with the abelian case, we move on to the case of a definably simple compact group. Using Proposition 4.28, it will be enough to prove the following restatement of [PP07, Lemma 4.5]. The original proof is based on the work in [BO04]. We, on the other hand, will use as a basis the theory we just developed for definably compact abelian groups.

Proposition 4.55. *Let $M_0 = \langle \mathbb{R}, <, +, \cdot, \dots \rangle$ be an o-minimal expansion of the real field, and $G(\mathbb{R})$ a connected compact Lie group definable over M_0. Let $\mathrm{st} : G \to G(\mathbb{R})$ be the standard part map, then $(G(\mathbb{R}), \mathrm{st})$ compactly dominates G. In particular, G is an fsg group.*

Proof. We first establish the following result on the Lebesgue measure.
<u>Claim:</u> Let $\mathcal{M}$ be a saturated expansion of M_0. Consider the box $T = [0, r)^n \subseteq \mathcal{M}^n$, where $r \in \mathbb{R}$. Let $\mathrm{st} : T \to T(\mathbb{R})$ be the standard part map. Then, for any two disjoint definable subsets $X, Y \subseteq T$, we have $\mathcal{L}(\mathrm{st}(X) \cap \mathrm{st}(Y)) = 0$, where $\mathcal{L}$ is the Lebesgue measure on $T(\mathbb{R})$.
<u>Proof:</u> Notice that $T = [0, r)^n \subseteq \mathcal{M}^n$, with addition modulus r, is a definably compact abelian group. Hence, by Proposition 4.54, it is fsg and therefore it is compactly dominated by T/T^{00}. Using for example Corollary 4.41, it is easy to see that T/T^{00} coincides with $T(\mathbb{R})$ and T is compactly dominated by $T(\mathbb{R})$ via the standard part map $\mathrm{st} : T \to T(\mathbb{R})$. This is equivalent to saying that, for any two disjoint subset $X, Y \subseteq T$, we have $\mathcal{L}(\mathrm{st}(X) \cap \mathrm{st}(Y)) = 0$, where $\mathcal{L}$ is the Lebesgue measure on $T(\mathbb{R})$. ■

Now, let m be the Haar measure of $G(\mathbb{R})$. To prove that $G(\mathbb{R})$ compactly dominates G via the standard part map, is suffices to show that for any two disjoint definable subsets $X, Y \subseteq G$, we have $m(\mathrm{st}(X) \cap \mathrm{st}(Y)) = 0$.

To this end, consider a definable atlas $\{U_i : i < n\}$ of G. As G is compact we can assume there is some $r \in \mathbb{R}$ such that, for every $i < n$, both $X \cap U_i$ and $Y \cap U_i$ are contained in the box $[0, r)^n$. By the claim, we know that $\mathcal{L}(\mathrm{st}(X \cap U_i) \cap \mathrm{st}(Y \cap U_i)) = 0$. Now, as $G(\mathbb{R})$ is a Lie group, its Haar measure m is non singular with respect to the Lebesgue measure $\mathcal{L}$. Therefore, we can also conclude $m(\mathrm{st}(X \cap U_i) \cap \mathrm{st}(Y \cap U_i)) = 0$. As this is true for all local charts U_i, we have $m(\mathrm{st}(X) \cap \mathrm{st}(Y)) = 0$, as we wanted. □

Finally, we will need the following to put all the pieces together.

Lemma 4.56. *Let G be a definable group and let $H < G$ be a normal subgroup.*

(i) If H and G/H are definably amenable, so is G.

(ii) If H and G/H have a finitely satisfiable global invariant measure, so do G.

Proof. (i) Let μ and λ be translation invariant global measures on H and G/H respectively. By Lemma 2.17, we can assume ν to be definable over some small model M. We define a measure η on G as follow. Let $X \subseteq G$ be a set definable over some small model $N \supseteq M$. For $gH \in G/H$, let $f(gH) = \mu(g^{-1}X \cap H)$. This is well defined by translation invariance of μ. Moreover it depends only on $\mathrm{tp}(gH/N)$ and the corresponding function $f : S_{G/H}(N) \to [0, 1]$ is continuous by definability of μ. Therefore, using a slight abuse of notation, we can write

$$\eta(X) = \int_{q \in S_{G/H}(N)} \mu(q^{-1}X \cap H)\, \mathrm{d}\lambda_q.$$

It is easy to see that this defines a translation invariant measure.

(ii) Use the same notation as before. Suppose that μ and λ are finitely satisfiable over M, and let X be a set definable over some model $N \supseteq M$. Suppose further that

$$\eta(X) = \int_{q \in S_{G/H}(N)} \mu(q^{-1}X \cap H)\, \mathrm{d}\lambda_q > 0.$$

Then there is some $\varepsilon > 0$ such that the set $A = \{q \in S_{G/H}(N) : f(q) > \varepsilon\}$ has positive λ-measure. Notice that, since μ is definable, A is open. Hence, there is a definable subset $A_0 \subseteq A$ such that $\lambda(A_0) > 0$. Using the fact that λ is finitely satisfiable over M, we can find some $gH \in A_0(M)$. In particular $gH \in A$, hence, by definition of A, we have $\mu(g^{-1}X \cap H) > \varepsilon$. Using the fact that μ is finitely satisfiable, we can then find some $h \in H(M)$ such that $h \in g^{-1}X \cap H$. In particular, we have $gh \in X(M)$. $\square$

We are now ready to prove our main theorem.

Theorem 4.57. *Definably compact groups in an o-minimal theory are fsg.*

Proof. We procede by induction on $\dim(G)$ using the decomposition theorems. We have the folllowing cases:

G **definably simple.** By Proposition 4.28, we can assume that G is a semialgebraic group defined over $\mathbb{R}_{alg}$. We can therefore conclude using Proposition 4.55 and the discussion after it.

G **definably semisimple.** By Lemma 4.56, it suffices to prove this for $Z(G)$ and for $G/Z(G)$. The former is an abelian group, hence we can conclude using Proposition 4.54. By Theorem 4.30, $G/Z(G)$ is the direct product of definably simple groups. Thus, we can conclude using the previous case and the fact that the direct product of fsg groups is an fsg group.

General case. If G is not a definably connected semisimple group, then G has an infinite normal commutative subgroup N. By Proposition 4.54, we know the theorem to be true for N. Thus, we may assume $G \neq N$. As $\dim(G/N) < \dim(G)$, by inductive hypothesis we know that G/N is fsg. As both N and G/N are fsg groups, G is also an fsg group. $\square$

As we have stated at the beginning of this section, compact domination for definably compact groups is equivalent to the previous statement. However, we would like to restate this fact for easier reference.

Theorem 4.58. *Definably compact groups in an o-minimal theory are compactly dominated.*

Proof. By the previous theorem, definably compact groups are fsg. Therefore, by Proposition 2.21, they have a generically stable measure. Moreover, since the theory is o-minimal, Theorem 4.48 tells us that this measure is smooth. We can then apply Theorem 2.37 to conclude that they are compactly dominated. $\square$

Bibliography

[Ber06] Alessandro Berarducci. O-minimal spectra, infinitesimal subgroups and cohomology. *The Journal of Symbolic Logic*, 72(4):21, 2006.

[BF09] Alessandro Berarducci and Antongiulio Fornasiero. O-minimal cohomology: finiteness and invariance results. *Journal of Mathematical Logic*, 9(02):167–182, 2009.

[BM11] Alessandro Berarducci and Marcello Mamino. On the homotopy type of definable groups in an o-minimal structure. *Journal of the London Mathematical Society*, 83(3):563–586, February 2011.

[BM13] Alessandro Berarducci and Marcello Mamino. Groups definable in two orthogonal sorts. *arXiv preprint arXiv:1304.1380*, pages 1–18, 2013.

[BO03] Alessandro Berarducci and Margarita Otero. Transfer methods for o-minimal topology. *Journal of Symbolic Logic*, 68(3):785–794, September 2003.

[BO04] Alessandro Berarducci and Margarita Otero. An additive measure in o-minimal expansions of fields. *Quarterly Journal of Mathematics*, 55(4):411–419, 2004.

[BOPP05] Alessandro Berarducci, Margarita Otero, Yaa'cov Peterzil, and Anand Pillay. A descending chain condition for groups definable in o-minimal structures. *Annals of Pure and Applied Logic*, 134(2-3):303–313, July 2005.

[Dol04] Alfred Dolich. Forking and independence in o-minimal theories. *The Journal of Symbolic Logic*, 69(1):pp. 215–240, 2004.

[Edm03] Mário J. Edmundo. Solvable groups definable in o-minimal structures. *Journal of Pure and Applied Algebra*, 185(1-3):103–145, December 2003.

[EO04] Mário J. Edmundo and Margarita Otero. Definably Compact Abelian Groups. *Journal of Mathematical Logic*, 04(02):163–180, December 2004.

[EW09] Mário J. Edmundo and Arthur Woerheide. The Lefschetz coincidence theorem in o-minimal expansions of fields. *Topology and its Applications*, 156(15):2470–2484, September 2009.

[HM13] K.H. Hofmann and S.A. Morris. *The Structure of Compact Groups: A Primer for the Student - A Handbook for the Expert.* De Gruyter Studies in Mathematics. De Gruyter, 2013.

[HP11] Ehud Hrushovski and Anand Pillay. On NIP and invariant measures. *Journal of the European Mathematical Society*, 13(4):1005–1061, 2011.

[HPP08] Ehud Hrushovski, Ya'acov Peterzil, and Anand Pillay. Groups, measures, and the NIP. *Journal of the American Mathematical Society*, 21(2):563–596, 2008.

[HPP11] Ehud Hrushovski, Ya'acov Peterzil, and Anand Pillay. On central extensions and definably compact groups in o-minimal structures. *Journal of Algebra*, 327:71–106, 2011.

[Joh14] Will Johnson. A pathological o-minimal quotient. *ArXive 1404.3175*, pages 1–7, 2014.

[Mar02] David Marker. *Model Theory : An Introduction.* Graduate Texts in Mathematics. Springer, 2002.

[OPP96] Margarita Otero, Ya'acov Peterzil, and Anand Pillay. On groups and rings definable in o-minimal expansions of real closed fields. *Bulletin of the London Mathematical Society*, 28(1):7–14, 1996.

[Pil88] Anand Pillay. On groups and fields definable in o-minimal structures. *Journal of Pure and Applied Algebra*, 53, 1988.

[Pil04] Anand Pillay. Type-definability, compact Lie groups, and o-minimality. *Journal of Mathematical Logic*, pages 1–19, 2004.

[PP07] Ya'acov Peterzil and Anand Pillay. Generic sets in definably compact groups. *Fundamenta Mathematicae*, 193(2):153–170, 2007.

[PPS00] Yaa'cov Peterzil, Anand Pillay, and Sergei Starchenko. Definably simple groups in o-minimal structures. *Transactions of the American Mathematical Society*, 352(10):4397–4419, 2000.

[PS86] Anand Pillay and Charles Steinhorn. Definable sets in ordered structures. i. *Transactions of the American Mathematical Society*, 295(2):pp. 565–592, 1986.

[Rob49] Julia Robinson. Definability and decision problems in arithmetic. *The Journal of Symbolic Logic*, 14(2):pp. 98–114, 1949.

[Sim10] Pierre Simon. Finding generically stable measures. pages 1–22, 2010.

[Sim14] Pierre Simon. A guide to NIP theories, 2014.

[Str94] Adam Strzebonski. Euler characteristic in semialgebraic and other o-minimal groups. *Journal of Pure and Applied Algebra*, 4049(93), 1994.

[vdD98] Lou van den Dries. *Tame Topology and O-minimal Structures.* 150 184. Cambridge University Press, 1998.

[vdDM94] Lou van den Dries and Chris Miller. On the real exponential field with restricted analytic functions. *Israel Journal of Mathematics*, 85(1-3):19–56, 1994.

[Wil96] Alex James Wilkie. Model completeness results for expansions of the ordered field of real numbers by restricted pfaffian functions and the exponential function. *Journal of the American Mathematical Society*, 9(4):pp. 1051–1094, 1996.

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